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TOTAL VARIATION REGULARIZATION FOR IMAGE DENOISING; I. GEOMETRIC THEORY.

WILLIAM K. ALLARD

Abstract. Let Ω be an open subset of $\mathbf{R}^n$ where $2 \le n \le 7$; we assume $n \le 2$ because the case $n = 1$ has been treated elsewhere (see [Alli]) and is quite different from the case $n > 1$; we assume $n \le 7$ is that our work will make use of the regularity theory for area minimizing hypersurfaces. Let

$$\mathcal{F}(\Omega) = \mathbf{L}_1(\Omega) \cap \mathbf{L}_\infty(\Omega)).$$

Suppose $s \in \mathcal{F}(\Omega)$ and Suppose

$$\gamma : \mathbb{R} \to [0, \infty)$$

is locally Lipschitzian, positive on $\mathbb{R} \sim \{0\}$ and zero at zero. Let

$$F(f) = \int_\Omega \gamma(|f(x) - s(x)|)\, d\mathcal{L}^n x \quad \text{for } f \in \mathcal{F}(\Omega);$$

here $\mathcal{L}^n$ is Lebesgue measure on $\mathbb{R}^n$. Note that $F(f) = 0$ if and only if $f(x) = s(x)$ for $\mathcal{L}^n$ almost all $x \in \mathbb{R}^n$. In the denoising literature F would be called a *fidelity* term in that it measures deviation from s which could be a noisy grayscale image. Let $\epsilon > 0$ and let

$$F_\epsilon(f) = \epsilon \mathbf{TV}(f) + F(f) \quad \text{for } f \in \mathcal{F}(\Omega);$$

here $\mathbf{TV}(f)$ is the total variation of f. A minimizer of F_ϵ is called a *total variation regularization of* s. Rudin, Osher and Fatemi and Chan and Esedoglu have studied total variation regularizations of F where $\gamma(y) = y^2$ and $\gamma(y) = |y|$, $y \in \mathbb{R}$, respectively.

Let f be a total variation regularization of F. The first main result of this paper is that the reduced boundaries of the sets $\{f \ge y\}$, $y \in \mathbf{R}$, are embedded $C^{1+\mu}$ hypersurfaces for any $\mu \in (0, 1)$ in case $n > 2$ and any $\mu \in (0, 1]$ in case $n = 2$; moreover, the generalized mean curvature of the sets $\{f \ge y\}$ will be bounded in terms of y, ϵ and the magnitude of $|s|$ near the point in question. In fact, this result holds for a rather general class of fidelities. A second result gives precise curvature information about the reduced boundary of $\{f \ge y\}$ near points where s is smooth provided F is convex. This curvature information will allow us to construct a number of interesting examples of total variation regularizations in this and in a subsequent paper.

In addition, a number of other theorems about regularizations are proved.

Contents

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1. Introduction and statement of main results.

Throughout this paper, n is an integer such that $2 \le n \le 7$, $\mathcal{L}^n$ is Lebesgue measure on $\mathbb{R}^n$ and Ω is an open subset of $\mathbb{R}^n$.

We require $n \ge 2$ because the problems we consider are very different in case $n = 1$; see [Alli]. We require $n \le 7$ because we will be using the regularity theory of mass minimizing integral currents in $\mathbb{R}^n$ of codimension one; as is well known, these currents are free of singularities when $n \le 7$ but may possess singularities if $n > 7$; see [FE, 5.4.15]. This work is motivated by image denoising applications in which it is often the case that $1 \le n \le 4$.

1.1. **Total variation.**

This work is based on the notion of the total variation of a locally summable function, which we now define.

Definition 1.1.1. *Suppose $f \in \mathbf{L}_1^{loc}(\Omega)$. Then $\mathbf{TV}(f, \cdot)$, the* **total variation** *of f, is the largest Borel regular measure on Ω such that, for any open subset U of Ω, $\mathbf{TV}(f, U)$ equals the supremum of*

$$\int_\Omega f \,\mathbf{div}\, X \, d\mathcal{L}^n$$

as X ranges over C^1 vector fields on Ω whose support is a compact subset of U and for which $|X(x)| \le 1$ whenever $x \in \Omega$.

In particular, if f is C^1 and B is a Borel subset of Ω then

$$\mathbf{TV}(f, B) = \int_B |\nabla f| \, d\mathcal{L}^n. \tag{1.1.1}$$

Moreover, if E a Lebesgue measurable subset of Ω with Lipschitz boundary then $\mathbf{TV}(E, B)$ equals the $(n-1)$ dimensional Hausdorff measure of the intersection of the boundary of E with B; here and in what follows we will frequently write "E" for "1_E" where 1_E is the indicator function of E.

Suppose $f \in \mathbf{L}_1^{loc}(\Omega)$. We say f is of **bounded variation on** Ω if $\mathbf{TV}(f, \Omega)$ is finite. If $\mathbf{TV}(f, \cdot)$ is a Radon measure on Ω which will be the case if and only if $\mathbf{TV}(f, K) < \infty$ whenever K is a compact subset of Ω we say f is of **locally bounded variation on** Ω. We let

$$\mathbf{BV}(\Omega) \quad \text{and} \quad \mathbf{BV}^{loc}(\Omega)$$

be the vector spaces of those $f \in \mathbf{L}_1(\Omega)$ which are of bounded variation on Ω and of locally bounded variation on Ω, respectively.

If E is a Lebesgue measurable subset of Ω the **perimeter of** E is, by definition,t $\mathbf{TV}(E, \Omega)$; we say E is of **locally finite perimeter** if $E \in \mathbf{BV}^{loc}(\Omega)$.

1.2. (ϵ, F) **-minimizers.**

Definition 1.2.1. *We let*

$$\mathcal{F}(\Omega) = \mathbf{L}_1(\Omega) \cap \mathbf{L}_\infty(\Omega)$$

with the topology induced from its inclusion in $\mathbf{L}_1(\Omega)$.

We let

$$\mathbf{F}(\Omega)$$

be the family of real valued functions on $\mathcal{F}(\Omega)$.

Suppose $F \in \mathbf{F}(\Omega)$ and $0 < \epsilon < \infty$. We let

$$\mathbf{m}_\epsilon(F)$$

be the set of $f \in \mathbf{BV}^{loc}(\Omega)$ such that for any compact subset K of Ω we have

$$\epsilon\mathbf{TV}(f, K) + F(f) \leq \epsilon\mathbf{TV}(g, K) + F(g)$$

*whenever $g \in \mathcal{F}(\Omega)$ and g is essentially equal to f in $\Omega \sim K$. We say a member of $\mathbf{m}_\epsilon(F)$ is a (ϵ, F)***-minimizer.**

It will be useful to extend the foregoing notions to functionals defined on sets, as follows.

Definition 1.2.2. *We let*

$$\mathcal{M}(\Omega)$$

be the family of Lebesgue measurable subsets D of Ω such that $\mathcal{L}^n(D) < \infty$ with the topology induced by its embedding in $\mathcal{F}(\Omega)$ via $\mathcal{M}(\Omega) \ni E \mapsto 1_E$. We let

$$\mathbf{M}(\Omega)$$

be the family of real valued functions on $\mathcal{M}(\Omega)$.

Suppose $M \in \mathbf{M}(\Omega)$ and $0 < \epsilon < \infty$. We let

$$\mathbf{n}_\epsilon(M)$$

be the set of $D \in \mathcal{M}(\Omega)$ with locally finite perimeter such that for any compact subset K of Ω we have

$$\epsilon\mathbf{TV}(D, K) + M(D) \leq \epsilon\mathbf{TV}(E, K) + M(E)$$

*whenever $E \in \mathcal{M}(\Omega)$ and E is essentially equal to D in $\Omega \sim K$. We say a member of $\mathbf{n}_\epsilon(M)$ is a (ϵ, M)***-minimizer***.*

1.3. **Denoising.** Suppose

$$s \in \mathcal{F}(\Omega);$$

s could be a grayscale representation of a degraded image which we wish to denoise. If $n = 2$ then Ω could be the computer screen. Suppose

$$\gamma : \mathbb{R} \to [0, \infty)$$

is locally Lipschitzian, positive on $\mathbb{R} \sim \{0\}$ and zero at zero. We define $F \in \mathbf{F}(\Omega)$ by letting

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^n x \quad \text{for } f \in \mathcal{F}(\Omega).$$

In the context of denoising F would be called a **fidelity**; this is because for each $\eta > 0$ there is $\delta > 0$ such that

$$F(f) < \delta \;\Rightarrow\; \int_\Omega |f - s|\, d\mathcal{L}^n < \eta \quad \text{whenever } f \in \mathcal{F}(\Omega).$$

If $0 < \epsilon < \infty$ the members of $\mathbf{m}_\epsilon(F)$ would be called **total variation regularizations of s (with respect to the fidelity F and smoothing parameter ϵ).**

For a very informative discussion of the use of total variation regularizations in the field of image processing see the Introduction of [CE]. We will not discuss image processing any further except to note that the notion of total variation regularization in image processing is useful for other purposes besides denoising.

Of particular interest is when $1 \le p < \infty$ and

$$\gamma(y) = \frac{1}{p}|y|^p \quad \text{whenever } y \in \mathbb{R}. \tag{1.3.1}$$

Rudin, Osher and Fatemi [ROF] studied the case $p = 2$ and Chan and Esedoglu [CE] studied the case $p = 1$.

The main goal of this paper is to state and prove theorems about the regularity and geometric properties of the sets $\{f \ge y\}$, $y \in \mathbb{R}$, when f is a minimizer of F_ϵ. We will find that the geometry of these sets is rather restricted. These results will allow us to construct an number of interesting examples of minimizers in [AW2], a sequel to this paper; we hope these examples will provide insights into the nature of total variation regularization. At the end of this paper we will determine $\mathbf{m}_\epsilon(F)$ when s is the indicator function of a square and γ is as in (1.3.1).

1.4. **The space $\mathcal{C}_\lambda(\Omega)$.** Suppose $0 \le \lambda < \infty$. For reason which will become clear shortly it will be useful to introduce

$$\mathcal{C}_\lambda(\Omega)$$

which, by definition, is the family of Lebesgue measurable subsets D of Ω such that

$$\mathbf{TV}(D, K) \le \mathbf{TV}(E, K) + \lambda \mathcal{L}^n((D \sim E) \cup (E \sim D))$$

whenever K is a compact subset of Ω, E is a Lebesgue measurable subset of Ω and E is essentially equal to D in $\Omega \sim K$. We now state a regularity theorem for $\mathcal{C}_\lambda(\Omega)$.

Theorem 1.4.1 (Regularity Theorem)**.** *Suppose $0 < \mu < \infty$ and $0 < \beta < 1$. There is θ such that $0 < \theta < 1$ and with the following property:*

Suppose

(i) *$0 \le \lambda < \infty$ and $D \in \mathcal{C}_\lambda(\Omega)$;*
(ii) *M is the support of the generalized gradient of the indicator function of D;*
(iii) *$a \in M$, $0 < R < \infty$ and $\{x \in \mathbb{R}^n : |x - a| < R\} \subset \Omega$;*
(iv) *$\lambda R \le \theta$, $r = \theta R$ and $B = \{x \in \mathbb{R}^n : |x - a| < r\}$.*

Then $M \cap B$ is an embedded hypersurface in Ω of class $C^{1+\mu}$; moreover, if N is a continuous field of unit normals to $M \cap B$ then

$$|N(x) - N(w)| \le \beta\,(|x - a|/r)^\mu \quad \textit{whenever } x, w \in M \cap B;$$

finally, if L is a line perpendicular to the tangent hyperplane to $M \cap B$ at a then L intersects $M \cap B$ in a most one point.

In case $n = 2$ we may take $\mu = 1$.

The proof of this Theorem is an exercise in the use of techniques from area minimization theory, the theory of functions of least gradient and geometric measure theory which have been in the literature for over thirty years.

The relevance of $\mathcal{C}_\lambda(\Omega)$ to image denoising is as follows.

Theorem 1.4.2. *Suppose s, γ and F are as in 1.3, $0 < \epsilon < \infty$, $f \in \mathbf{m}_\epsilon(F)$ and $y \in \mathbb{R}$. Then*

$$\{f \ge y\} \in \mathcal{C}_\lambda(\Omega). \tag{1.4.1}$$

where λ is the Lipschitz constant of γ on $[\mathbf{ess\ inf}(f - s), \mathbf{ess\ sup}\,(f - s)]$ divided by ϵ.

See 6.2 for the proof.

1.5. **Localizing with respect to the value.** *For the remainder of this Introduction we suppose s, γ, F are as in 1.3 and we suppose γ, and therefore F, is convex.*

For each $y \in \mathbb{R}$ we define
$$L_y,\ U_y \in \mathbf{M}(\Omega)$$
by letting
$$L_y(E) = \liminf_{z \to y} \frac{F(z1_E) - F(y1_E)}{z - y} \quad \text{and} \quad U_y(E) = \limsup_{z \to y} \frac{F(z1_E) - F(y1_E)}{z - y};$$
here 1_E is the indicator function of E. It is a simple matter to verify that L_y and U_y are finite; see 7. Evidently, $L_y \le U_y$ for $y \in \mathbb{R}$. We will show later that $L_y = U_y$ for all but countable many $y \in \mathbb{R}$.

Theorem 1.5.1. *Suppose $f \in \mathbf{m}_\epsilon(F)$ and $y \in \mathbb{R} \sim \{0\}$. Then*
$$\{f < y\} \in \mathbf{n}_\epsilon(-L_y) \quad \textit{and} \quad \{f \le y\} \in \mathbf{n}_\epsilon(-U_y) \quad \textit{if } y < 0$$
and
$$\{f \ge y\} \in \mathbf{n}_\epsilon(L_y) \quad \textit{and} \quad \{f > y\} \in \mathbf{n}_\epsilon(U_y) \quad \textit{if } y > 0.$$

In fact, this Theorem holds for a class functionals F somewhat more general than those specified above.

A sort of converse to the preceding Theorem is as follows.

Theorem 1.5.2. *Suppose G is a $\mathcal{L}^n \times \mathcal{L}^1$ measurable subset of $\Omega \times \mathbb{R}$ such that*
$$G \times (\Omega \times [0,\infty)) \quad \textit{and} \quad (\Omega \times (-\infty, 0)) \sim G$$
have finite $\mathcal{L}^n \times \mathcal{L}^1$ measure; for $\mathcal{L}^1$ almost all $y \in \mathbb{R} \sim \{0\}$, $\{x \in \Omega : (x,y) \in G\} \in \mathbf{n}_\epsilon(U_y)$ if $y > 0$ and $\{x \in \Omega : (x,y) \notin G\} \in \mathbf{n}_\epsilon(-U_y)$ if $y < 0$; and $f \in \mathcal{F}(\Omega)$ is such that
$$f = \int_0^\infty 1_{\{x:(x,y)\in G\}}\, d\mathcal{L}^1 y - \int_{-\infty}^0 1_{\{x:(x,y)\notin G\}}\, d\mathcal{L}^1 y.$$
Then $f \in \mathbf{m}_\epsilon(F)$.

This result is of particular interest when $\gamma(y) = |y|$ for $y \in \mathbb{R}$; see 9.2.

1.6. **Results on curvature.** In the light of Theorem 1.5.1 we are motivated to study $\mathbf{m}_\epsilon(M)$ where $M \in \mathbf{M}(\Omega)$.

Theorem 1.6.1. *Suppose*

(i) *$M \in \mathbf{M}(\Omega)$, $\zeta \in \mathbf{L}_\infty(\Omega)$ and*
$$M(E) = \int_E \zeta\, d\mathcal{L}^n \quad \textit{whenever } E \in \mathcal{M}(\Omega);$$
(ii) *U is an open subset of Ω, j is a nonnegative integer, $0 < \mu < 1$; and $\zeta|U$ is of class $C^{j+\mu}$;*
(iii) *$0 < \epsilon < \infty$, $D \in \mathbf{n}_\epsilon(M)$ and M is the intersection of U with the support of the generalized gradient of the indicator function of D.*

Then M is and embedded hypersurface of U of class $C^{j+2+\mu}$ and
$$H(x) = -\frac{1}{\epsilon}\zeta(x)N(x) \quad \textit{for } x \in M \tag{1.6.1}$$

where H is the mean curvature vector of M and N is the outward pointing unit normal along M to the support of generalized function corresponding to the indicator function of D.

Moreover, if ζ is of class C^1 on U and Q is the square of the length of the second fundamental form of M as defined in 3 then

$$\int_M \epsilon\left(|\nabla_M \phi(x)|^2 + \phi(x)^2 Q(x)\right) - \phi(x)^2 \nabla\zeta(x) \bullet N(x)\, d\mathcal{H}^{n-1}x \geq 0 \tag{1.6.2}$$

for any $\phi \in \mathcal{D}(\Omega)$; here, for each $x \in M$, $\nabla_M \phi(x)$ is the orthogonal projection of $\nabla\phi(x)$ on $\mathbf{Tan}(M, x)$ and Q is the square of the length of the second fundamental form of M.

See Section 3 for the relevant definitions. This Theorem will apply in the context of denoising if s as in 1.3 is sufficiently regular in U.

Every one of these results will be used in our determination of minimizers.

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2. Some basic notations and conventions.

We find the mathematical infrastructure of normal and integral currents to be indispensable in carrying out this work. For that reason we will adopt, for the most part, the notation and terminology of [FE]; note the extensive glossary, list of notations and index starting on page 669 of that book. We avoided using that notation and terminology in the Introduction in order to make it more accessible to readers not familiar with [FE].

We let

$$\mathbf{N} \quad \text{and} \quad \mathbf{P}$$

be the set of nonnegative integers and the set of positive integers, respectively.

Whenever $a \in \mathbb{R}^n$ and $0 < r < \infty$ we let

$$\mathbf{U}^n(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\} \quad \text{and} \quad \mathbf{B}^n(a, r) = \{x \in \mathbb{R}^n : |x - a| \leq r\}.$$

We let

$$\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

We let

$$\mathbf{e}_1, \ldots, \mathbf{e}_n \quad \text{and} \quad \mathbf{e}^1, \ldots, \mathbf{e}^n$$

be the standard basis vectors and covectors for $\mathbb{R}^n$ and its dual space, respectively. We let

$$\mathbf{E}^n = \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n \in \bigwedge^n \mathbb{R}^n$$

be the standard orientation on $\mathbb{R}^n$.

We let

$$\mathbf{int}, \quad \mathbf{cl}, \quad \text{and} \quad \mathbf{bdry}$$

stand for "interior", "closure" and "boundary", respectively.

Whenever $A \subset \mathbb{R}^n$ and a is an accumulation point of A we let

$$\mathbf{Tan}(A, a) = \bigcap_{0<r<\infty} \mathbf{cl}\,\{t(x - a) : 0 < t < \infty \text{ and } x \in A \cap (\mathbf{B}^n(a, r) \sim \{a\})\}$$

and we let

$$\mathbf{Nor}(A,a) = \bigcap_{w \in \mathbf{Tan}(A,a)} \{v \in \mathbb{R}^n : v \bullet w \le 0\}.$$

We let

$$\mathcal{H}^m, \ m \in [0,\infty),$$

m dimensional Hausdorff measure on $\mathbb{R}^n$.

Whenever A, D, E are Lebesgue measurable subsets of Ω we let

$$\Sigma_A(D,E) = \mathcal{L}^n(A \cap ((D \sim E) \cup (E \sim D))) = \int_A |1_D - 1_E| \, d\mathcal{L}^n;$$

and note that

$$\Sigma_A(\Omega \sim D, \Omega \sim E) = \Sigma_A(D,E). \tag{2.0.1}$$

Note also that for any $\mathcal{L}^n$ measurable subset A of Ω

$$\mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \ni (D,E) \mapsto \Sigma_A(D,E)$$

is a pseudometric on $\mathcal{M}(\Omega)$.

Whenever f is a function mapping a subset of a normed vector space into another normed vector space, a is an interior point of the domain of f and f is Fréchet differentiable at a we let

$$\partial f(a)$$

be the Fréchet differential of f at a.

If V is a vector space, $v \in V$ and ψ belongs to the dual space of V we frequently write

$$< v, \psi > \quad \text{instead of} \quad \psi(v).$$

Whenever $E \subset \Omega$ we let 1_E, the **indicator function of** E, be the function on Ω which is 1 on E and 0 on $\Omega \sim E$. We will often write "E" instead of "1_E"; for example, in what follows, we will often write "$E \in \mathcal{F}(\Omega)$", "$E \in \mathbf{BV}(\Omega)$", "$[E]$" and "$||\partial[E]||$" instead of "$1_E \in \mathcal{F}(\Omega)$", "$1_E \in \mathbf{BV}(\Omega)$", "$[1_E]$" and "$||\partial[1_E]||$", respectively.

We let

$$\mathcal{X}(\Omega)$$

be the vector space of smooth compactly supported vector fields on Ω.

We use **spt** as an abbreviation for "support"; so, for example, if $X \in \mathcal{X}(\Omega)$,

$$\mathbf{spt}\, X = \mathbf{cl}\, \{x \in \Omega : X(x) \neq 0\}.$$

Whenever $y, z \in \mathbb{R}$ we let

$$y \vee z = \max\{y,z\}, \quad \text{we let} \quad y \wedge z = \min\{y,z\}$$

and we note that $y + z = y \vee z + y \wedge z$.

3. Second fundamental forms and mean curvature.

Suppose $m \in \mathbf{P}$, $m < n$ and M is an embedded m dimensional submanifold of class C^2 in $\mathbb{R}^n$.

The **second fundamental form of** M is the function Π on M whose value at $a \in M$ is a linear map from $\mathbf{Nor}(M, a)$ into the symmetric linear maps from $\mathbf{Tan}(M, a)$ to itself characterized by the requirement that if U is an open subset of $\mathbb{R}^n$, $a \in U \cap M$; $N : U \to \mathbb{R}^n$; N is of class C^1; and $N(x) \in \mathbf{Nor}(M, x)$ whenever $x \in U \cap M$ then

$$\Pi(a)(N(a))(v) \bullet w = \partial N(a)(v) \bullet w \quad \text{for } v, w \in \mathbf{Tan}(M, a)$$

The **mean curvature vector of** M is, by definition, the function on H on M whose value at a point a of M is that member $H(a)$ of $\mathbf{Nor}(M, a)$ such that

$$H(a) \bullet u = \operatorname{trace} \Pi(a)(u) \quad \text{whenever } u \in \mathbf{Nor}(M, a);$$

in the classical literature the mean curvature vector is $1/m$ times H; hence the word "mean". It turns out the factor $1/m$ in is inconvenient when one is working, as we will be, with the first variation of area and for this reason we omit it. The direction of the mean curvature vector, and not just its magnitude, will be important in this work.

The **length of the second fundamental form of** M is, by definition, the function on M whose value at the point a of M equals

$$\left(\sum_{j=1}^{n-m} \sum_{i=1}^{m} |\Pi(a)(u_j)(u_i)|^2 \right)^{1/2}$$

whenever $u_1, \ldots, u_n$ is an orthonormal basis for $\mathbb{R}^n$ such that $u_1, \ldots, u_m \in \mathbf{Tan}(M, a)$.

Suppose $f : \Omega \to \mathbb{R}$ is C^2; $\nabla f(x) \neq 0$ whenever $x \in \Omega$; y is a point of the range of f; and $M = \{f = y\}$. Let Π be the second fundamental form of M and let H be the mean curvature vector of M. It follows that if $a \in M$ then

$$\Pi(a)(\nabla f(a))(u) \bullet v = \partial(\nabla f)(a)(u) \bullet v \quad \text{whenever } u, v \in \mathbf{Tan}(M, a).$$

For example, let $\Omega = \mathbb{R}^n \sim \{0\}$, let $f(x) = |x|^{/}2$ for $x \in \Omega$, suppose $0 < R < \infty$ and let $M = \{x \in \mathbb{R}^n : |x| = R\}$. Then $\nabla f(x) = x$ for $x \in \Omega$. It follows that if $a \in M$ then

$$\Pi(a)(a)(v) \bullet w = \frac{v \bullet w}{|a|} \quad \text{whenever } v, w \in \mathbf{Tan}(M, a),$$

$$H(a) = \frac{n-1}{R^2} a$$

and the length $\Pi(a)$ equals the square root of $(n-1)/R^2$.

4. Some basic notions of geometric measure theory.

4.1. **Spaces of smooth functions and their duals.** Suppose Y is a Banach space. We let

$$\mathcal{E}(\Omega, Y), \quad \mathcal{E}'(\Omega, Y), \quad \mathcal{D}(\Omega, Y), \quad \mathcal{D}'(\Omega, Y)$$

be the space of smooth Y valued functions on Ω with the strong topology as described in [FE, 4.1.1]; the space of continuous real valued linear functions on $\mathcal{E}(\Omega, Y)$ with the weak topology as described in [FE, 4.1.1]; the space of compactly supported members of $\mathcal{E}(\Omega, Y)$ with the strong topology as described in [FE, 4.1.1]; and the space of continuous real valued linear functions on $\mathcal{D}(\Omega, Y)$ with the weak topology as described in [FE, 4.1.1], respectively.

We identify each member of T of $\mathcal{E}'(\Omega, Y)$ with its restriction to $\mathcal{D}(\Omega, Y)$ which is a member of $\mathcal{D}'(\Omega, Y)$.

We let

$$\mathcal{E}(\Omega), \quad \mathcal{E}'(\Omega), \quad \mathcal{D}(\Omega), \quad \mathcal{D}'(\Omega)$$

equal $\mathcal{E}(\Omega, \mathbf{R})$, $\mathcal{E}'(\Omega, \mathbf{R})$, $\mathcal{D}(\Omega, \mathbf{R})$ and $\mathcal{D}'(\Omega, \mathbf{R})$, respectively.

Thus

$$\mathcal{X}(\Omega) = \mathcal{D}(\Omega, \mathbb{R}^n).$$

4.2. **Currents.** For each $m \in \mathbf{N}$ we let

$$\mathcal{E}^m(\Omega); \quad \mathcal{E}_m(\Omega); \quad \mathcal{D}^m(\Omega); \quad \mathcal{D}_m(\Omega)$$

be $\mathcal{E}(\Omega, Y)$; $\mathcal{E}'(\Omega, Y)$; $\mathcal{D}(\Omega, Y)$; and $\mathcal{D}'(\Omega, Y)$, respectively, with $Y = \bigwedge^m \mathbb{R}^n$. Thus $\mathcal{D}_m(\Omega)$ is the space of m dimensional **currents** on Ω and $\mathcal{E}_m(\Omega)$ is the space of m dimensional currents with compact support on Ω. We define the **boundary operator**

$$\partial : \mathcal{D}_m(\Omega) \to \mathcal{D}_{m-1}(\Omega)$$

by setting $\partial T(\omega) = T(d\omega)$ whenever $T \in \mathcal{D}_m(\Omega)$ and $\omega \in \mathcal{D}_{m-1}(\Omega)$; here d is exterior differentiation.

We let

$$\mathbf{V}^n \in \mathcal{D}(\Omega) \tag{4.2.1}$$

be such that $\mathbf{V}^n(x) = \mathbf{E}^n$ for $x \in \Omega$.

Suppose $T \in \mathcal{D}_m(\Omega)$. As in [FE, 4.1.5] we let

$$||T||,$$

the **total variation measure of** T, be the largest Borel regular measure on Ω such that

$$||T||(G) = \sup\{|T(\omega)| : \omega \in \mathcal{D}^m(\Omega),\ ||\omega|| \le 1 \text{ and } \mathbf{spt}\,\omega \subset G\}$$

for each open subset G of Ω; here $||\cdot||$ is the **comass** which in case $m \in \{0, 1, n-1, n\}$ is the Euclidean norm; these are the only cases we will encounter in this paper. It follows immediately from this definition that

$$||T||(G) \le \liminf_{\nu\to\infty} ||S_\nu||(G) \quad \text{for any open subset } G \text{ of } \Omega \tag{4.2.2}$$

whenever S is a sequence in $\mathcal{D}_m(\Omega)$ such that $S_\nu(\omega) \to T(\omega)$ as $\nu \to \infty$ whenever $\omega \in \mathcal{D}^m(\Omega)$. We let

$$\mathbf{M}(T) = ||T||(\Omega)$$

and call this nonnegative extended real number the **mass** of T. We say T is **representable by integration** if $||T||$ is a Radon measure which is equivalent to the statement that $||T||(K) < \infty$ whenever K is a compact subset of Ω. If this is the case and $\overrightarrow{T}$ is the $||T||$ measurable function with values in $\{\xi \in \bigwedge_m \mathbb{R}^n : ||\xi|| = 1\}$ defined in [FE, 4.1.7] there is a unique extension of T to the $||T||$ summable functions on Ω with values in $\bigwedge^m \mathbb{R}^n$, which we continue to denote by T, such that

$$T(\omega) = \int < \overrightarrow{T}(x), \omega(x) > d||T||x$$

whenever ω is a $||T||$ summable function on Ω with values in $\bigwedge^m \mathbb{R}^n$. If $T \in \mathcal{D}_m(\Omega)$ is representable by integration and η is a bounded Borel function on Ω with values in $\bigwedge^l \mathbb{R}^n$, $l \in \mathbf{N}$ and $l \leq m$ we let

$$T \llcorner \eta \in \mathcal{D}_{m-l}(\Omega)$$

be such that

$$T \llcorner \eta(\omega) = \int < \overrightarrow{T}(x), (\eta \wedge \omega)(x) > \, d||T||x \quad \text{for } \omega \in \mathcal{D}^{m-l}(\Omega).$$

4.3. **Mapping currents.** Whenever $T \in \mathcal{D}_m(\Omega)$ and F is a smooth map from Ω to the open subset Γ of some Euclidean space whose restriction to the support of T is proper we let

$$F_\# T \in \mathcal{D}_m(\Gamma)$$

be such that $F_\# T(\omega) = T(F^\# \omega)$ for any $\omega \in \mathcal{D}^m(\Gamma)$; here the **pullback** $F^\#$ is as in [FE, 4.1.6]. If F carries Ω diffeomorphically onto Γ, T is representable by integration and $\overrightarrow{T}(x)$ is decomposable for $||T||$ almost all $x \in \Omega$ we have

(4.3.1) $$\int \omega(y) \, d||F_\# T||y = \int \omega(F(x)) \left| \bigwedge\nolimits_m \partial F(x)(\overrightarrow{T}(x)) \right| \, d||T||x$$

for nonnegative Borel function ω on Γ.

4.4. **Slicing.** Suppose $m, l \in \mathbf{P}$, $m \geq l$, T is a locally flat m-dimensional current in $\mathbb{R}^m$ as defined in [FE, 4.1.12] and $f : \Omega \to \mathbb{R}^l$ is locally Lipschitzian. Note that if both T and ∂T are representable by integration then T is locally flat; this will always be the case when we apply slicing in this paper. For $y \in \mathbb{R}^l$ we follow [FE, 4.3.1] and define

$$< T, f, y >$$

the **slice of** T **in** $f^{-1}[\{y\}]$ to be that member of $\mathcal{D}_{m-l}(\Omega)$ which, if it exists, satisfies

$$< T, f, y > (\psi) = \lim_{r \downarrow 0} \frac{T \llcorner [f^\# (\mathbf{B}^l(y,r) \wedge \mathbf{V}^l)](\psi)}{\mathcal{L}^l(\mathbf{B}^l(y,r))} \quad \text{whenever } \psi \in \mathcal{D}^{m-l}(\Omega)$$

where $T \llcorner [f^\# (\mathbf{B}^l(y,r) \wedge \mathbf{V}^l)]$ is defined as in [FE, 4.3.1]. Then, by [FE, 4.3.1], $< T, f, y >$ exists for $\mathcal{L}^l$ almost all y and satisfies

(4.4.1) $$\mathbf{spt} < T, f, y > \subset f^{-1}[\{y\}] \quad \text{and} \quad \partial < T, f, y > = (-1)^l < \partial T, f, y > .$$

Moreover, we have from [FE, 4.3.2] that

(4.4.2) $$\int \Phi(y) < T, f, y > (\psi) \, d\mathcal{L}^l y = [T \llcorner f^\# (\Phi \wedge \mathbf{V}^l)](\psi)$$

whenever Φ is a bounded Borel function on $\mathbb{R}^l$ and $\psi \in \mathcal{D}^{m-l}(\Omega)$ and that

(4.4.3) $$\int \left(\int v || < T, f, y > || \right) d\mathcal{L}^l y = \int v \, d||T \llcorner f^\# \mathbf{V}^l]||$$

whenever v is a nonnegative Borel function on Ω.

Of particular interest to us will be the case when $l = 1$. Suppose $u : \Omega \to \mathbb{R}$ is locally Lipschitzian and both T and ∂T are representable by integration. From [FE, 4.2.1] and [FE, 4.3.4] we obtain

$$\begin{aligned} < T, u, r > &= (\partial T) \llcorner \{u > r\} - \partial(T \llcorner \{u > r\} \\ &= \partial(T \llcorner \{u \le r\}) - (\partial T) \llcorner \{u \le r\} \\ &= \partial(T \llcorner \{u < r\}) - (\partial T) \llcorner \{u < r\} \\ &= (\partial T) \llcorner \{u \ge r\}) - (\partial T \llcorner \{u \ge r\} \end{aligned} \tag{4.4.4}$$

for $\mathcal{L}^1$ almost all r. If now $S \in \mathcal{D}_m(\Omega)$ and both S and ∂S are representable by integration we infer from (4.4.4) that

$$\begin{aligned} &\partial(S \llcorner \{u \le r\} + T \llcorner \{u > r\}) \\ &\qquad =< S - T, u, r > +(\partial S) \llcorner \{u \le r\} + (\partial T) \llcorner \{u > r\} \end{aligned} \tag{4.4.5}$$

for $\mathcal{L}^1$ almost all r. It follows that

$$\begin{aligned} &\int_a^b ||\partial(S \llcorner \{u \le r\} + T \llcorner \{u > r\}) - \partial S||(\{u \le r\})\, d\mathcal{L}^1 r \\ &\qquad \le \mathbf{Lip}(u|\{a < u < b\})||S - T||(\{a < u < b\}) \end{aligned} \tag{4.4.6}$$

whenever $-\infty < a < b < \infty$.

4.5. **The current corresponding to a locally summable function.**

Definition 4.5.1. *Whenever $f \in \mathbf{L}_1^{loc}(\Omega)$ we let*

$$[f] \in \mathcal{D}_n(\Omega)$$

be defined by

$$[f](\phi \mathbf{V}^n) = \int_\Omega \phi f \, d\mathcal{L}^n \quad \textit{whenever } \phi \in \mathcal{D}(\Omega).$$

Suppose $f \in \mathbf{L}_1^{loc}(\Omega)$. For any $X \in \mathcal{X}(\Omega)$ we have

$$d(X \lrcorner \mathbf{V}^n) = \operatorname{div} X \tag{4.5.1}$$

so that

$$\partial[f](X \lrcorner \mathbf{V}^n) = \int f \operatorname{div} X \, d\mathcal{L}^n. \tag{4.5.2}$$

It follows that

$$||\partial[f]||(B) = \mathbf{TV}(f, B) \quad \text{whenever } B \text{ is a Borel subset of } \Omega. \tag{4.5.3}$$

Thus, in the terminology of [FE, 4.1.7], it follows from [FE, 4.5.9] that $\{[f] : f \in \mathbf{BV}(\Omega)\}$ is the vector space of n dimensional normal currents in Ω and $\{[f] : f \in \mathbf{BV}^{loc}(\Omega)\}$ is the vector space of locally normal n dimensional currents in Ω.

For any $f \in \mathbf{L}_1^{loc}(\Omega)$ and any $y \in \mathbb{R}$ we have

$$[\{f \ge y\}] + [\{f < y\}] = [\Omega].$$

Thus if $f, g \in \mathbf{L}_1^{loc}(\Omega)$ we have

$$[\{f \ge y\}] - [\{g \ge y\}] = [\{g < y\}] - [\{f < y\}]. \tag{4.5.4}$$

Also, if $f, g \in \mathbf{L}_1^{loc}(\Omega)$ and $y \in \mathbb{R}$ then

$$\mathbf{spt}\,[\{f \ge y\}] - [\{g \ge y\}] = \mathbf{spt}\,[\{g < y\}] - [\{f < y\}] \subset \mathbf{spt}\,[f - g]. \tag{4.5.5}$$

Finally, if E is a Lebesgue measurable subset of Ω we have $[E]+[\Omega \sim E]=[\Omega]$ and $\partial[\Omega]=0$ so

$$\partial[\Omega \sim E] = -\partial[E]. \tag{4.5.6}$$

4.6. **A mapping formula.**

Theorem 4.6.1. *Suppose Γ is an open subset of $\mathbb{R}^n$; $f \in \mathbf{L}_1^{loc}(\Omega)$; $F : \Omega \to \Gamma$ is locally Lipschitzian; the restriction of F to the support of $[f]$ is proper; A is the set of $y \in \Gamma$ such that $F^{-1}[\{y\}]$ is finite and such that if $F(x)=y$ then F is differentiable at x; and $g : \Gamma \to \mathbb{R}$ is such that*

$$g(y) = \begin{cases} \sum_{x \in F^{-1}[\{y\}]} f(x) \operatorname{sgn} \det \partial F(x) & \text{if } y \in A, \\ 0 & \text{else;} \end{cases}$$

Then $g \in \mathbf{L}_1^{loc}(\Gamma)$ and

$$F_\#[f] = [g]. \tag{4.6.1}$$

In particular, if F is univalent and $\det \partial F(x) > 0$ *for $\mathcal{L}^n$ almost all $x \in \Omega$ then*

$$F_\#[f] = [f \circ F^{-1}].$$

Proof. See [FE, 4.1.25]. □

4.7. **Densities and density ratios.** Suppose μ is a measure on Ω, $m \in \mathbf{N}$ and

$$\alpha(m) = \mathcal{L}^m(\{x \in \mathbb{R}^m : |x| < 1\}).$$

For each $a \in \Omega$ we set

$$\Theta^m(\mu, a, r) = \frac{\mu(\mathbf{B}(a,r))}{\alpha(m) r^m} \quad \text{whenever } 0 < r < \mathbf{dist}\,(a, \mathbb{R}^n \sim \Omega)$$

and we set

$$\Theta^m(\mu, a) = \lim_{r \to 0} \Theta^m(\mu, a, r)$$

provided this limit exists.

4.8. **Sets of finite perimeter.** Suppose E has **locally finite perimeter** which means, by definition, that $E \in \mathbf{BV}^{loc}(\Omega)$. Proceeding as in [FE, 4.5.5], we say $u \in \mathbf{S}^{n-1}$ is an **exterior normal to E at** $b \in \Omega$ if

$$\Theta^n(\{x \in E : (x-b) \bullet u > 0\} \cup \{x \in \Omega \sim E : (x-b) \bullet u < 0\}, b) = 0$$

We let

$$\mathbf{n}_E$$

be the set of $(b,u) \in \Omega \times (\{0\} \cup \mathbf{S}^{n-1})$ such that *either* u is an exterior normal to E at b *or* $u=0$ and there is no exterior normal to E at b; note that $\mathbf{n}_E$ is a function with domain Ω. We let

$$\mathbf{b}(E),$$

the **reduced boundary of** E, equal to the set of points $b \in \Omega$ such that $\mathbf{n}_E(b) \in \mathbf{S}^{n-1}$.

Theorem 4.8.1. *Suppose E is a subset of Ω with locally finite perimeter. The following statement hold:*

(i) $\mathbf{b}(E)$ *is a Borel set which is is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable;*
(ii) $||\partial[E]|| = \mathcal{H}^{n-1} \llcorner \mathbf{b}(E)$;

(iii) *for $\mathcal{H}^{n-1}$ almost all $b \in \mathbf{b}(E)$ we have*

$$*\mathbf{n}_E(b) = \overrightarrow{\partial[E]}(b) \quad \text{and} \quad \Theta^{n-1}(||\partial[E]||, b) = 1;$$

here $*$ is the **Hodge star operator** *as defined in* [FE, 1.7.8].

(iv) *for $\mathcal{H}^{n-1}$ almost all $b \in \Omega \sim \mathbf{b}(E)$, $\Theta^{n-1}(||\partial[E]||, b) = 0$ and*

$$\text{either } \Theta^n(\mathcal{L}^n \llcorner E, b) = 0 \text{ or } \Theta^n(\mathcal{L}^n \llcorner (\Omega \sim E), b) = 0.$$

Proof. See [FE, 4.5.6]. □

It follows that

(4.8.1) $$\partial[E](X \lrcorner \mathbf{V}^n) = \int X \bullet \mathbf{n}_E \, d||\partial[E]|| \quad \text{whenever } X \in \mathcal{X}(\Omega).$$

Theorem 4.8.2. *Suppose $E \in \mathcal{M}(\mathbb{R}^n)$ and C is a closed convex subset of $\mathbb{R}^n$. Then*

$$\mathbf{M}(\partial[C \cap E]) \leq \mathbf{M}(\partial[E]).$$

Proof. We may assume that E has finite perimeter since otherwise the Theorem holds trivially. Whenever $0 < r < \infty$ we let $E_r = E \cap \mathbf{U}^n(0, r)$.

Suppose $0 < r < \infty$. Let $\rho : \mathbb{R}^n \to C$ be such that $|x - \rho(x)| = \mathbf{dist}\,(x, C)$ for $x \in \mathbb{R}^n$. Since $\mathbf{spt}\,[E_r]$ is compact we infer from Theorem 4.6.1 that

$$[C \cap E_r] = \rho_\#[E_r]$$

so that, as $\mathbf{Lip}\,\rho \leq 1$,

$$\mathbf{M}(\partial[C \cap E_r]) \leq \mathbf{M}(\partial[E_r])$$

with equality only if $[E] = [C \cap E]$.

It follows from (4.4.6) that

$$\int_R^S \mathbf{M}(\partial[E_r] - \partial[E]) \, d\mathcal{L}^1 r \leq \int_R^S ||\partial[E]||(\mathbb{R}^n \sim \mathbf{U}^n(0, r)) \, d\mathcal{L}^1 r + \mathcal{L}^n(E \cap (\mathbf{U}^n(0, S) \sim \mathbf{U}^n(0, R))$$

whenever $0 < R < S < \infty$. Thus there is a sequence s in $(0, \infty)$ with limit ∞ such that

$$\lim_{\nu \to \infty} \mathbf{M}(\partial[E] - \partial[E_{s_\nu}]) = 0.$$

From (4.2.2) we infer that

$$\mathbf{M}(\partial[E \cap C]) \leq \liminf_{\nu \to \infty} \mathbf{M}(\partial[E_{s_\nu} \cap C]) \leq \liminf_{\nu \to \infty} \mathbf{M}(\partial[E_{s_\nu}]) = \mathbf{M}(\partial[E]).$$

□

4.9. **Basic facts about functions of bounded variation.** Suppose $f \in \mathbf{BV}^{loc}(\Omega)$; then

(4.9.1) $$\partial[f](\omega) = \int \partial[\{f \geq y\}](\omega) \, d\mathcal{L}^1 y \quad \text{whenever } \omega \in \mathcal{D}^{n-1}(\Omega)$$

and

(4.9.2) $$||\partial[f]|| = \int ||\partial[\{f \geq y\}]|| \, d\mathcal{L}^1 y.$$

See, for example, [FE, 4.5.9(13)]. These formulae are absolutely fundamental for this work.

We endow $\mathbf{L}_1^{loc}(\Omega)$ with the topology induced by the seminorms

$$\mathbf{L}_1^{loc}(\Omega) \ni f \mapsto \int_K |f| \, d\mathcal{L}^n$$

corresponding to compact subsets K of Ω.

The following two well known theorems may be proved using regularization.

Theorem 4.9.1 (Approximation Theorem)**.** *Suppose* $f \in \mathbf{L}_1^{loc}(\Omega)$. *Then* $f \in \mathbf{BV}^{loc}(\Omega)$ *if and only if there is a sequence* g *in* $\mathcal{E}(\Omega)$ *such that*

(i) $g_\nu \to f$ *in* $\mathbf{L}_1^{loc}(\Omega)$ *as* $\nu \to \infty$;

(ii) $||\partial[g_\nu]|| \to ||\partial[f]||$ *weakly as* $\nu \to \infty$.

Theorem 4.9.2 (Compactness Theorem)**.** *Suppose* C *is a sequence of nonnegative real numbers and* K *is a sequence of compact subsets of such that* $\cup_{\nu=0}^{\infty} K_\nu = \Omega$. *Then*

$$\bigcap_{\nu=0}^{\infty} \left\{ f \in \mathbf{BV}^{loc}(\Omega) : \int_{K_\nu} |f| \, d\mathcal{L}^n + ||\partial[f]||(K_\nu) \leq C_\nu \right\}$$

is a compact subset of $\mathbf{L}_1^{loc}(\Omega)$.

Theorem 4.9.3. *Suppose* $f \in \mathbf{BV}^{loc}(\Omega)$ *and* $y \in \mathbf{R}$. *Then* $f \wedge y, f \vee y \in \mathbf{BV}^{loc}(\Omega)$ *and*

$$||\partial[f \wedge y]|| + ||\partial[f \vee y]|| = ||\partial[f]||. \tag{4.9.3}$$

Proof. It is trivial that the right hand side of (4.9.3) does not exceed the left hand side of (4.9.3).

We consider only the case $f \geq 0$ and leave to the reader the straightforward extension of our argument to the general case. One readily shows that

$$[f \wedge y](\omega) = \int_0^y [\{f \geq y\}](\omega) \, d\mathcal{L}^1 y \quad \text{and} \quad [f \vee y](\omega) = \int_y^\infty [\{f \geq y\}](\omega) \, d\mathcal{L}^1 y$$

whenever $\omega \in \mathcal{D}^n(\Omega)$. Applying ∂ one infers

$$||\partial[f \wedge y]|| \leq \int_0^y ||\partial[\{f \geq y\}]|| \, d\mathcal{L}^1 y \quad \text{and} \quad ||\partial[f \vee y]|| \leq \int_y^\infty ||\partial[\{f \geq y\}]|| \, d\mathcal{L}^1 y.$$

By (4.9.1) the sum of the right hand sides of these inequalities is $||\partial[f]||$. Thus the left hand side of (4.9.3) does not exceed the right hand side. □

4.10. **"Layer cake" formulae.** These elementary formulae will be very useful in this work.

Proposition 4.10.1. *Suppose* f, g *and* ϕ *are real valued Lebesgue measurable functions on* Ω *and* $\phi \geq 0$. *Then*

$$\int_{\{f<g\}} \phi(g-f) \, d\mathcal{L}^n = \int_{-\infty}^{\infty} \left(\int_{\{g \geq y\} \sim \{f \geq y\}} \phi \, d\mathcal{L}^n \right) d\mathcal{L}^1 y. \tag{4.10.1}$$

Proof. From Tonelli's Theorem we infer that

$$\begin{aligned}\int_{\{f<g\}} \phi(g-f)\, d\mathcal{L}^n &= \int_{\{f<g\}} \phi(x) \left(\int_{[f(x),g(x))} d\mathcal{L}^1 \right) d\mathcal{L}^n x \\ &= \int_{\{(x,y)\in\Omega\times\mathbb{R}: f(x)<y\le g(x)\}} \phi\, d(\mathcal{L}^n \times \mathcal{L}^1) \\ &= \int_{-\infty}^{\infty} \left(\int_{\{g\ge y\}\sim\{f\ge y\}} \phi\, d\mathcal{L}^n \right) d\mathcal{L}^1 y.\end{aligned}$$

□

Chan and Esedoglu in [CE] call the following elementary formula the "layer cake" formula; it is indispensable in this work.

Corollary 4.10.1. *Suppose f, g are real valued Lebesgue measurable functions on Ω. Then*

$$\int_\Omega |f-g|\, d\mathcal{L}^n = \int_{-\infty}^{\infty} \Sigma_\Omega(\{f \ge y\}, \{g \ge y\})\, d\mathcal{L}^1 y. \tag{4.10.2}$$

Proof. Let $\phi(x) = 1$ for $x \in \Omega$. Applying the preceding Proposition twice we obtain

$$\int_{\{f<g\}} (g-f)\, d\mathcal{L}^n = \int_{-\infty}^{\infty} \mathcal{L}^n(\{g \ge y\} \sim \{f \ge y\})\, d\mathcal{L}^1 y$$

and

$$\int_{\{g<f\}} (f-g)\, d\mathcal{L}^n = \int_{-\infty}^{\infty} \mathcal{L}^n(\{f \ge y\} \sim \{g \ge y\})\, d\mathcal{L}^1 y.$$

Now add. □

4.11. **The class $\mathcal{G}(\Omega)$.** Let

$$p : \Omega \times \mathbb{R} \to \Omega \quad \text{and} \quad q : \Omega \times \mathbb{R} \to \mathbb{R}$$

carry $(x, y) \in \Omega \times \mathbb{R}$ to x and y, respectively.

Whenever G is an $\mathcal{L}^n \times \mathcal{L}^1$ measurable subset of $\Omega \times \mathbb{R}$ we let

$$[G] \in \mathcal{D}_{n+1}(\Omega \times \mathbb{R})$$

be as in 4.2.1 with $\mathbf{V}^n$ there replaced by $(p^\# \mathbf{V}^n) \wedge dq$; that is,

$$[G](\psi(p^\# \mathbf{V}^n) \wedge dq) = \int_G \psi\, d(\mathcal{L}^n \times \mathcal{L}^1) \quad \text{whenever } \psi \in \mathcal{D}(\Omega \times \mathbb{R}).$$

Whenever $G \subset \Omega \times \mathbb{R}$ we let

$$G^+ = (\Omega \times [0, \infty)) \cap G \quad \text{and we let} \quad G^- = (\Omega \times (-\infty, 0)) \sim G.$$

Proposition 4.11.1. *Suppose G is an $\mathcal{L}^n \times \mathcal{L}^1$ measurable subset of $\Omega \times \mathbb{R}$. Then*

$$[G] = [G^+] - [G^-] + [\Omega \times (-\infty, 0)].$$

Proof. We have

$$[G] = [G \cap (\Omega \times [0, \infty))] + [G \cap (\Omega \times (-\infty, 0))]$$

and

$$[\Omega \times (-\infty, 0)] = [G \cap (\Omega \times (-\infty, 0))] + [(\Omega \times (-\infty, 0)) \sim G].$$

Since $G \cap (\Omega \times [0, \infty)) = G^+$ and $(\Omega \times (-\infty, 0)) \sim G = G^-$ the Proposition follows. □

Definition 4.11.1. *We let*

$$\mathcal{G}(\Omega)$$

be the family of Lebesgue measurable subsets G of $\Omega \times \mathbb{R}$ such that

$$(\mathcal{L}^n \times \mathcal{L}^1)(G^+ \cup G^-) < \infty$$

and

$$q[\mathbf{spt}\,[G^+] \cup \mathbf{spt}\,[G^-]] \text{ is compact.}$$

Note that if $G \in \mathcal{G}(\Omega)$ then for $\mathcal{L}^1$ almost all y we have

$$\{x : (x,y) \in G^+\} \in \mathcal{M}(\Omega) \quad \text{and} \quad \{x : (x,y) \in G^-\} \in \mathcal{M}(\Omega).$$

Definition 4.11.2. *Whenever $G \in \mathcal{G}(\Omega)$ we let*

$$G^{\downarrow} : \Omega \to \mathbb{R}$$

be such that

$$G^{\downarrow}(x) = \mathcal{L}^1(\{y : (x,y) \in G^+\}) - \mathcal{L}^1(\{y : (x,y) \in G^-\})$$

if both $\{y : (x,y) \in G^+\}$ and $\{y : (x,y) \in G^-\}$ belong to $\mathcal{M}(\mathbb{R})$ and such that $G^{\downarrow}(x) = 0$ otherwise.

Note that $G^{\downarrow} \in \mathcal{F}(\Omega)$ and $(\mathcal{L}^n \times \mathcal{L}^1)(G) = \int_\Omega |G^{\downarrow}|\, d\mathcal{L}^n$.

Definition 4.11.3. *Whenever $f : \Omega \to \mathbb{R}$ we let*

$$f^{\uparrow} = \{(x,y) \in \Omega \times \mathbb{R} : f(x) \geq y\}.$$

Suppose $f : \Omega \to \mathbb{R}$. Evidently,

$$f \in \mathcal{F}(\Omega) \;\Leftrightarrow\; f^{\uparrow} \in \mathcal{G}(\Omega).$$

Fubini's Theorem implies that

$$[(f^{\uparrow})^{\downarrow}] = [f] \quad \text{whenever } f \in \mathcal{F}(\Omega).$$

Proposition 4.11.2. *Suppose $G \in \mathcal{G}(\Omega)$, $\phi \in \mathcal{D}(\Omega)$ and $\Psi \in \mathcal{E}(\Omega)$. Then*

$$\begin{aligned}
&p_{\#}\left(\partial([G^+] - [G^-]) \llcorner \Psi \circ q\right)(\phi \mathbf{V^n}) \\
&\quad = (-1)^n([G^+] - [G^-])\left(p^{\#}(\phi \mathbf{V}^n) \wedge (\Psi' \circ q) dq\right) \\
&\quad = (-1)^n \int_\Omega \phi(x) \left(\int_{\{y:(x,y)\in G^+\}} \Psi'\, d\mathcal{L}^1 - \int_{\{x:(x,y)\in G^-\}} \Psi'\, d\mathcal{L}^1 \right) d\mathcal{L}^n x.
\end{aligned} \tag{4.11.1}$$

Proof. The first equation follows from the fact that

$$d((\Psi \circ q) \wedge p^{\#}(\phi \mathbf{V}^n)) = (\Psi' \circ q) dq \wedge p^{\#}(\phi \mathbf{V}^n)$$

and the second follows from Fubini's Theorem. □

Proposition 4.11.3. *Suppose $G \in \mathcal{G}(\Omega)$. Then*

$$[G^{\downarrow}] = (-1)^n p_{\#}\left((\partial[G^+] - [G^-]) \llcorner q\right)$$

and

$$\partial[G^{\downarrow}] = (-1)^{n+1} p_{\#}(\partial([G^+] - [G^-]) \llcorner dq).$$

Proof. Letting $\Psi(y) = y$ for $y \in \mathbb{R}$ in the preceding Proposition we deduce the first equation; the second equation is an immediate consequence of the first. □

Corollary 4.11.1. *Suppose $G \in \mathcal{G}(\Omega)$. Then $p|\mathbf{spt}\,\partial[G]$ is proper and*
$$\partial[G^{\downarrow}] = (-1)^{n+1} p_{\#}(\partial([G]) \llcorner dq).$$

Proof. Since $\mathbf{spt}\,\partial[\Omega \times (-\infty] = \Omega$ and since $(\partial[\Omega \times (-\infty])\llcorner dq = 0$ the Corollary follows immediately from Proposition 4.11.1 and the preceding Proposition. □

Proposition 4.11.4. *Suppose $G \in \mathcal{G}(\Omega)$ and $\partial[G]$ is representable by integration. Then*
$$||\partial[G^{\downarrow}]||(B) \leq \int ||\partial[\{x : (x,y) \in G\}]||(B)\, d\mathcal{L}^1 y$$
for any Borel subset B of Ω.

Proof. Suppose U is an open subset of Ω, $\omega \in \mathcal{D}^{n-1}(\Omega)$, $\mathbf{spt}\,\omega \subset U$ and $|\omega| \leq 1$.

For each $y \in \mathbb{R}$ let $i_y(x) = (x,y)$ for $x \in \Omega$. From [FE, 4.3.8] we have
$$< [G], q, y >= i_{y\#}[\{x : (x,y) \in G\}] \quad \text{for } \mathcal{L}^1 \text{ almost all } y.$$
From Corollary 4.11.1, (4.4.2) and (4.4.1) we find that
$$\begin{aligned}(-1)^{n+1}\partial[G^{\downarrow}](\omega)| &= (\partial[G]\llcorner dq)(p^{\#}\omega)\\ &= \int < \partial[G], q, y > (p^{\#}\omega)\, d\mathcal{L}^1 y\\ &= -\int \partial[\{x : (x,y) \in G\}](\omega)\, d\mathcal{L}^1 y\\ &\leq \int ||\partial[\{x : (x,y) \in G\}]||(U)\, d\mathcal{L}^1 y\end{aligned}$$
from which the inequality to be proved immediately follows. □

We will find the following elementary Proposition to be useful.

Proposition 4.11.5. *Suppose $G \in \mathcal{G}(\Omega)$. The following statements are equivalent.*

(i) *For $\mathcal{L}^2$ almost all $(y,z) \in \mathbb{R}^2$ such that $y < z$ we have*
$$\mathcal{L}^n(\{x : (x,z) \in G \text{ and } (x,y) \notin G\}) = 0.$$
(ii) *For $\mathcal{L}^n$ almost all $x \in \Omega$ we have*
$$\mathcal{L}^2(\{(y,z) \in \mathbb{R}^2 : y < z,\ (x,z) \in G \text{ and } (x,y) \notin G\}) = 0.$$
(iii) *$[\{G^{\downarrow} \geq y\}] = [\{x : (x,y) \in G\}]$ for $\mathcal{L}^1$ almost all y;*
(iv) *$[\{y : (x,y) \in G\}] = [\{y : -\infty < y \leq G^{\uparrow}(x)\}]$ for $\mathcal{L}^n$ almost all $x \in \Omega$.*
(v) *$[G] = [(G^{\downarrow})^{\uparrow}]$.*

Proof. (i) and (ii) are equivalent by Tonelli's Theorem. Since
$$(G^{\downarrow})^{\uparrow} = \{(x,y) : G^{\downarrow}(x) \geq y\}$$
we find that (iii), (iv) and (v) are equivalent by Tonelli's Theorem.

Suppose (iii) holds. Since $\{G^{\downarrow} \geq z\} \subset \{G^{\downarrow} \geq y\}$ whenever $(y,z) \in \mathbb{R}^2$ and $y < z$ we find that (i) holds.

Suppose (ii) holds. By Tonelli's Theorem for $\mathcal{L}^n \times \mathcal{L}^1$ almost all $(x,z) \in G$ we have
$$(x,y) \in G \text{ for } \mathcal{L}^n \text{ almost all } y \in (-\infty, z).$$
Moreover, since $(\mathcal{L}^n \times \mathcal{L}^1)(G^+ \cup G^-) < \infty$ we infer from Tonelli's Theorem that
$$\mathcal{L}^1(G \cap (\{x\} \times \mathbb{R})) > 0 \quad \text{for } \mathcal{L}^n \text{ almost all } x \in \Omega.$$

Thus there is $f : \Omega \to \mathbb{R}$ such that

$$[\{y : (x, y) \in G\}] = (-\infty, f(x)) \quad \text{for } \mathcal{L}^n \text{ almost all } x \in \Omega.$$

It follows that $G^{\downarrow}(x) = f(x)$ for $\mathcal{L}^n$ almost all $x \in \Omega$ so (iv) holds. □

4.12. **Deformations and variations.**

Definition 4.12.1. *We let*

$$\mathcal{V}(\Omega)$$

be the set of ordered triples (I, h, K) such that

(i) *I is an open interval and $0 \in I$;*
(ii) *$h : I \times \Omega \to \Omega$ and h is smooth;*
(iii) *$h(0, x) = x$ for $x \in \Omega$;*
(iv) *$\Omega \ni x \mapsto h(t, x)$ carries Ω diffeomorphically onto itself for each $t \in I$;*
(v) *$K = \mathbf{cl}\,\{x \in \Omega : h(t, x) \neq x$ for some $t \in I\}$ is a compact subset of Ω.*

Whenever $(I, h, K) \in \mathcal{V}(\Omega)$ and $(t, x) \in I \times \Omega$ we let

$$h_t(x) = h(t, x), \quad \dot{h}_t(x) = \frac{d}{dt}h(t, x), \quad \ddot{h}_t(x) = \left(\frac{d}{dt}\right)^2 h(t, x).$$

Note that if $X \in \mathcal{X}(\Omega)$ and $h(t, x) = x + tX(x)$ for $(t, x) \in \mathbf{R} \times \Omega$ it is elementary that there exists an I and K such that $(I, h|(I \times \Omega), K) \in \mathcal{V}(\Omega)$.

Theorem 4.12.1. *Suppose*

(i) *$(I, h, K) \in \mathcal{V}(\Omega)$;*
(ii) *D is a subset of Ω with locally finite perimeter and finite Lebesgue measure and*

$$E_t = \{h_t(x) : x \in D\} \quad \text{whenever } t \in I.$$

(iii) *$A(t) = ||\partial[E_t]||(K)$ for each $t \in I$.*

Then A is smooth,

$$\dot{A}(0) = \int A_1 \, d||\partial[D]|| \quad \text{and} \quad \ddot{A}(0) = \int A_2 \, d||\partial[D]||$$

where, for each $x \in \mathbf{b}(D)$,

$$P(x) \text{ is orthogonal projection on } \{v \in \mathbb{R}^n : v \bullet \mathbf{n}_D(x) = 0\};$$

(4.12.1)

$$\begin{aligned}
a_1(x) &= P(x) \circ \partial\dot{h}_0(x) \circ P(x), \\
a_2(x) &= P(x)^{\perp} \circ \partial\dot{h}_0(x) \circ P(x), \\
a_3(x) &= P(x) \circ \partial\ddot{h}_0(x) \circ P(x), \\
A_1(x) &= \operatorname{trace} a_1(x), \\
A_2(x) &= (\operatorname{trace} a_1(x))^2 + \operatorname{trace}(a_2(x)^* \circ a_2(x) - a_1(x) \circ a_1(x)) + \operatorname{trace} a_3(x).
\end{aligned}$$

Proof. It follows from (4.6.1) that $[E_t] = h_{t\#}[D]$ and therefore $\partial[E_t] = h_{t\#}\partial[D]$ for any $t \in I$. It follows from (4.3.1) that

$$\frac{d}{dt}||\partial[E_t]||(K) = \int \frac{d}{dt}\left|\bigwedge\nolimits_{n-1} \partial h_t(x)(*\mathbf{n}_D(x))\right| d||\partial[D]||x$$

for any $t \in I$. That

$$\frac{d}{dt}\left|\bigwedge\nolimits_{n-1} \partial h_t(x)(*\mathbf{n}_D(x))\right|\Big|_{t=0} = A_1(x)$$
$$\left(\frac{d}{dt}\right)^2\left|\bigwedge\nolimits_{n-1} \partial h_t(x)(*\mathbf{n}_D(x))\right|\Big|_{t=0} = A_2(x)$$

are elementary calculations which may be found in [FE, 5.1.8]. □

Proposition 4.12.1. *Suppose*

(i) $(I, h, K) \in \mathcal{V}(\Omega)$
(ii) *D is a subset of Ω with locally finite perimeter and*

$$E_t = \{h_t(x) : x \in D\} \quad \textit{whenever } t \in I;$$

(iii) $\phi \in \mathcal{D}(\Omega)$.

Then

$$([E_t] - [D])(\phi\mathbf{V}^n) = \int_0^t \left(\int \phi(h_\tau(x))\, W_\tau(x)\, d||\partial[D]||x\right) d\mathcal{L}^1\tau$$

where, for each $t \in I$, we have set

$$W_t(x) = < \dot{h}_t(x) \wedge \bigwedge\nolimits_{n-1} \partial h_t(x)(*\mathbf{n}_D(x)), \mathbf{E} > \quad \textit{for } x \in \mathbf{b}(D).$$

Proof. For each $t \in I$ let $J_t = [0, t] \in \mathcal{D}_1(\mathbb{R})$ as in [FE, 4.1.8]. From [FE, 4.1.8] we have $||J_t \times \partial[D]|| = ||J_t|| \times ||\partial[D]||$ for each $t \in I$. From [FE, 4.1.8] and (iii) of Theorem 4.8.1 we have

$$\overrightarrow{J_t \times \partial[D]}(\tau, x) = (1, 0) \wedge \overrightarrow{\partial[D]}(x) = (1, 0) \wedge *\mathbf{n}_D(x)$$

whenever $(\tau, x) \in (0, t) \times \mathbf{b}(D)$.

Suppose $t \in I$. We obtain

$$[E_t] - [D] = h_{t\#}[D] - [D] = h_\#(J_t \times \partial[D])$$

from the homotopy formula of [FE, 4.1.9]; thus

$$\begin{aligned} &([E_t] - [D])(\phi\mathbf{V}^n) \\ &\quad = (J_t \times \partial[D])((\phi \circ h)h^\#\mathbf{V}^n) \\ &\quad = \int_0^t \left(\int \phi(h_\tau(x))\, W_\tau(x)\, d||\partial[D]||x\right) d\mathcal{L}^1\tau, \end{aligned}$$

as desired. □

Theorem 4.12.2. *Suppose*

(i) $(I, h, K) \in \mathcal{V}(\Omega)$*;*
(ii) *D is a subset of Ω with locally finite perimeter and finite Lebesgue measure and*

$$E_t = \{h_t(x) : x \in D\} \quad \textit{whenever } t \in I.$$

(iii) $\zeta \in \mathbf{L}_\infty(\Omega)$*;*
(iv) $B(t) = \int_{E_t} \zeta\, d\mathcal{L}^n$ *whenever $t \in I$.*

If ζ is continuous then B is continuously differentiable and

$$\dot{B}(0) = \int \zeta(\dot{h}_0 \bullet \mathbf{n}_D)\, d\|\partial[D]\|. \tag{4.12.2}$$

If ζ is continuously differentiable then B is twice continuously differentiable and

$$\ddot{B}(0) = \int \big(\zeta b + (\nabla\zeta \bullet \dot{h}_0)\dot{h}_0\big) \bullet \mathbf{n}_D)\, d\|\partial[D]\| \tag{4.12.3}$$

where, with P, a_1, a_2 as in Theorem 4.12.1, we have set

$$b(x) = \ddot{h}_0(x) + \operatorname{trace} a_1(x)\dot{h}_0(x) - a_2(x)(\dot{h}_0(x)) \quad \textit{for } x \in \mathbf{b}(D). \tag{4.12.4}$$

Proof. Let us assume for the moment that ζ is smooth. From Proposition 4.12.1 we infer that

$$B(t) = B(0) + \int_0^t \left(\int \zeta(h_\tau(x))\, W_\tau(x)\, d\|\partial[D]\|x \right) d\mathcal{L}^1\tau$$

where we have set

$$\xi_t(x) = \bigwedge\nolimits_{n-1} \partial h_t(x)(*\mathbf{n}_D(x)) \quad \text{and} \quad W_t(x) =< \dot{h}_t(x) \wedge \xi_t(x), \mathbf{E} >$$

for each $t, x) \in I \times \mathbf{b}(D)$.

Suppose $x \in \mathbf{b}(D)$. Let $u_1, \ldots, u_n$ be be an orthonormal sequence of vectors in $\mathbb{R}^n$ such that $\mathbf{n}_D(x) = u_1$ and $*u_n = u_2 \wedge \cdots \wedge u_n$; this implies

$$< w \wedge u_2 \wedge \cdots \wedge u_n, \mathbf{E}^n >= w \bullet u_1 < u_1 \wedge *u_1, \mathbf{E}^n >= w \bullet u_1 \quad \text{for any } w \in \mathbb{R}^n$$

and

$$\xi_t(x) = \bigwedge\nolimits_{n-1} \partial h_t(x)(u_2 \wedge \cdots \wedge u_n) \quad \text{whenever } t \in I;$$

see [FE, 1.7.8] for the properties of $*$.

It should now be clear that (4.12.2) holds in case ζ is smooth.

Let $u^1, \ldots, u^n$ be the sequence of covectors dual to $u_1, \ldots, u_n$. For each $i \in \{2, \ldots, n\}$ let $v_i = P(x)(\partial\dot{h}_0(x)(u_i))$ and let $c_i =< \partial\dot{h}_0(x)(u_i), u^1 >$. We have

$$\begin{aligned}
&\dot{h}_0(x) \wedge \frac{d}{dt}\xi_t(x)\big|_{t=0} \\
&\quad = \dot{h}_0(x) \wedge \sum_{i=2}^n \partial\dot{h}_0(x)(u_i) \wedge \big(u^i \lrcorner \xi_0(x)\big) \\
&\quad = \dot{h}_0(x) \wedge \sum_{i=2}^n v_i \wedge \big(u^i \lrcorner \xi_0(x)\big) + \dot{h}_0(x) \wedge \sum_{i=2}^n c_i u_1 \wedge \big(u^i \lrcorner \xi_0(x)\big) \\
&\quad = \left(\dot{h}_0(x) \bullet u_1 \left(\sum_{i=2}^n v_i \bullet u_i \right) - \sum_{i=2}^n (\dot{h}_0(x) \bullet u_i) c_i \right) u_1 \wedge \xi_0(x) \\
&\quad = \Big(\operatorname{trace} a_1(x)\dot{h}_0(x) \bullet u_1 - a_2(x)(\dot{h}_0(x)) \Big)\, u_1 \wedge \xi_0(x)
\end{aligned}$$

and

$$\ddot{h}_0(x) \wedge \xi_0(x) = (\ddot{h}_0(x) \bullet u_1) u_1 \wedge \xi_0(x).$$

It follows that

$$\frac{d}{dt}\zeta(h_t(x))W_t(x)\big|_{t=0} = \big((\nabla\zeta(x) \bullet \dot{h}_0(x))\dot{h}_0(x) + \zeta(x)b(x) \big) \bullet u_1$$

and we may conclude that (4.12.3) holds in case ζ is smooth.

To prove the Theorem holds in full generality we need only approximate ζ by smooth functions. $\square$

5. The spaces $\mathcal{B}_\lambda(\Omega)$ and $\mathcal{C}_\lambda(\Omega)$, $0 \le \lambda < \infty$.

We suppose throughout this section that

$$0 \le \lambda < \infty.$$

The spaces $\mathcal{B}_\lambda(\Omega)$ and $\mathcal{C}_\lambda(\Omega)$, which we now define, will be indispensable in this work.

5.1. **The definitions.**

Definition 5.1.1. *Whenever $f \in \mathbf{L}_1^{loc}(\Omega)$ and K is a compact subset of Ω we let*

$$\mathbf{c}(f, K)$$

be the set of $g \in \mathbf{L}_1^{loc}(\Omega)$ such that

$$\mathcal{L}^n(\{f \ne g\} \cap (\Omega \sim K)) = 0$$

and

$$\textbf{ess inf} f \le g(x) \le \textbf{ess sup} f \quad \text{for } \mathcal{L}^n \text{ almost all } x \in \Omega.$$

$$\mathcal{B}_\lambda(\Omega)$$

be the set of those $f \in \mathbf{BV}^{loc}(\Omega)$ such that for each compact subset K of Ω we have

$$||\partial[f]||(K) \le ||\partial[g]||(K) + \int_\Omega |f - g| \, d\mathcal{L}^n \quad \text{whenever } g \in \mathbf{c}(f, K).$$

Definition 5.1.2. *We let*

$$\mathcal{C}_\lambda(\Omega)$$

be the set of those subsets D of Ω with locally finite perimeter such that such that for each compact subset K we have

$$||\partial[D]||(K) \le ||\partial[E]||(K) + \lambda \Sigma_\Omega(D, E)$$

whenever $E \in \mathcal{M}(\Omega)$ and

$$\Sigma_{\Omega \sim K}(D, E) = 0.$$

Note that $\Sigma_{\Omega \sim K}(D, E) = 0$ if and only if $1_E \in \mathbf{c}(D, K)$.

5.2. **Basic theory of $\mathcal{B}_\lambda(\Omega)$ and $\mathcal{C}_\lambda(\Omega)$, $0 \le \lambda < \infty$.** As a consequence of (4.9.1) and (4.10.2) we find that

$$(5.2.1) \quad |\partial[g]||(K) + \int_\Omega |f - g| = \int_{-\infty}^{\infty} ||\partial[\{g \ge y\}]||(K) + \Sigma_\Omega(\{f \ge y\}, \{g \ge y\}) \, d\mathcal{L}^1 y$$

whenever $f, g \in \mathbf{BV}^{loc}(\Omega)$ and K is a compact subset of Ω.

Proposition 5.2.1. *Suppose $D \subset \Omega$. Then*

$$D \in \mathcal{C}_\lambda(\Omega) \;\Leftrightarrow \Omega \sim D \in \mathcal{C}_\lambda(\Omega).$$

Proof. This follows easily from (2.0.1) and (4.5.6); we leave the details to the reader. $\square$

The relationship between $\mathcal{B}_\lambda(\Omega)$ and $\mathcal{C}_\lambda(\Omega)$ is as follows.

Theorem 5.2.1.

$$\mathcal{C}_\lambda(\Omega) = \{E : E \subset \Omega \text{ and } 1_E \in \mathcal{B}_\lambda(\Omega)\}.$$

Proof. Suppose E is a subset of Ω with locally finite perimeter. It is trivial that $E \in \mathcal{C}_\lambda(\Omega)$ if $1_E \in \mathcal{B}_\lambda(\Omega)$. So suppose $E \in \mathcal{C}_\lambda(\Omega)$, K is a compact subset of Ω and $\mathbf{spt}\,[1_E - g] \subset K$. From (4.5.5) we find that, whenever $0 < y < 1$, $\Sigma_K(E, \{g \geq y\}) = 0$ so

$$||\partial[E]|| \leq ||\partial[\{g \geq y\}]||(K) + \lambda\Sigma_\Omega(E, \{g \geq y\}).$$

It follows from (4.9.1) and (4.10.2) that

$$\begin{aligned} ||\partial[E]||(K) &= \int_0^1 ||\partial[E]||(K)\, d\mathcal{L}^1 y \\ &\leq \int_0^1 ||\partial[\{g \geq y\}]||(K) + \lambda\Sigma_\Omega(E, \{g \geq y\})\, d\mathcal{L}^1 y \\ &\leq ||\partial[g]||(K) + \lambda \int_\Omega |1_E - g|\ d\mathcal{L}^n. \end{aligned}$$

□

The following Theorem is an elementary corollary of 5.3.1.

Theorem 5.2.2. *Suppose D is an open subset of Ω with smooth boundary. If $D \in \mathcal{C}_\lambda(\Omega)$ then*

$$|H| \leq \lambda$$

where H is the mean curvature vector of **bdry** D.

The converse of this statement is false as one sees in case $\lambda = 0$ by considering unstable minimal surfaces.

A good deal of what follows uses the ideas of [BDG].

Theorem 5.2.3. *Suppose $\lambda \in [0, \infty)$, $f \in \mathcal{B}_\lambda(\Omega)$ and $y \in \mathbf{R}$. Then*

$$\{f + y, yf, f \wedge y, f \vee y\} \subset \mathcal{B}_\lambda(\Omega).$$

Proof. Suppose $g \in \mathbf{BV}^{loc}(\Omega)$, K is a compact subset of Ω and $g \in \mathbf{c}(f + y, K)$. Then $g - y \in \mathbf{BV}^{loc}(\Omega) \cap \mathbf{c}(f, K)$ so

$$\begin{aligned} ||\partial[f + y]||(K) &= ||\partial[f]||(K) \\ &\leq ||\partial[g - y]||(K) + \lambda \int_K |f - (g - y)|\, d\mathcal{L}^n \\ &= ||\partial[g]||(K) + \lambda \int_K |(f + y) - g)|\, d\mathcal{L}^n \end{aligned}$$

so $f + y \in \mathcal{B}_\lambda(\Omega)$.

It is trivial that $0f = 0 \in \mathcal{B}_\lambda(\Omega)$.

Suppose $g \in \mathbf{BV}^{loc}(\Omega)$, K is a compact subset of Ω and $g \in \mathbf{c}(-f, K)$. Then $-g \in \mathbf{BV}^{loc}(\Omega) \cap \mathbf{c}(f, K)$ and and $\mathbf{spt}\,[f - (-g)] \subset K$ so

$$\begin{aligned} ||\partial[-f]||(K) &= ||\partial[f]||(K) \\ &\leq ||\partial[-g]||(K) + \lambda \int_K |f - (-g)|\, d\mathcal{L}^n \\ &= ||\partial[g]||(K) + \lambda \int_K |(-f) - g)|\, d\mathcal{L}^n \end{aligned}$$

so $-f \in \mathcal{B}_\lambda(\Omega)$.

Suppose $y > 0$, $g \in \mathbf{BV}^{loc}(\Omega)$ K is a compact subset of Ω and $g \in \mathbf{c}(yf, K)$. Then $g/y \in \mathbf{BV}^{loc}(\Omega) \cap \mathbf{c}(f, K)$ and $\mathbf{spt}\,[f - (g/y)] \subset K$ so

$$\begin{aligned}||\partial[yf]||(K) &= y||\partial[f]||(K) \\ &\le y\left(||\partial[g/y]||(K) + \lambda \int_K |f - (g/y)|\, d\mathcal{L}^n\right) \\ &= ||\partial[g]||(K) + \lambda \int_K |(yf) - g)|\, d\mathcal{L}^n\end{aligned}$$

so $yf \in \mathcal{B}_\lambda(\Omega)$.

If $y < 0$ we have

$$yf = (-y)(-f) \in \mathcal{B}_\lambda(\Omega)$$

by the results of the preceding two paragraphs,

If $y \le \mathbf{ess\ inf}\, f$ then $[f \wedge y] = [y]$ and $y \in \mathcal{B}_\lambda(\Omega)$ so $f \wedge y \in \mathcal{B}_\lambda(\Omega)$. If $\mathbf{ess\ sup}\, f \le y$ then $[f \wedge y] = [f] \in \mathcal{B}_\lambda(K)$ so $f \wedge y \in \mathcal{B}_\lambda(\Omega)$. So let us assume that

$$\mathbf{ess\ inf}\, f < y < \mathbf{ess\ sup}\, f.$$

Suppose $g \in \mathbf{BV}^{loc}(\Omega)$, K is a compact subset of Ω and $g \in \mathbf{c}(f \wedge y, K)$ Let $h = g + (f \vee y) - y$. For $\mathcal{L}^n$ almost $x \in \{f \le y\}$ we have $h(x) = g(x)$ so

$$\mathbf{ess\ inf}\, f = \mathbf{ess\ inf}\, f \wedge y \le g(x) = h(x)$$

and

$$h(x) = g(x) \le \mathbf{ess\ sup}\, f \wedge y \le \mathbf{ess\ sup}\, f.$$

For $\mathcal{L}^n$ almost $x \in \{f > y\}$ we have $h(x) = g(x) + f(x) - y$ so

$$\mathbf{ess\ inf}\, f = \mathbf{ess\ inf}\, f \wedge y + \mathbf{ess\ inf}\, f - y \le g(x) + f(x) - y = h(x)$$

and

$$h(x) \le \mathbf{ess\ sup}\, f \wedge y + f(x) - y = f(x) \le \mathbf{ess\ sup}\, f.$$

Moreover,

$$f - h = (f \wedge y + f \vee y - y) - (g + f \vee y - y) = f \wedge y - g.$$

It follows that $h \in \mathbf{c}(f, K)$ so

$$\begin{aligned}&||\partial[f \wedge y]||(K) + ||\partial[f \vee y]||(K) \\ &\qquad = ||\partial[f]||(K) \\ &\qquad \le ||\partial[h]||(K) + \lambda \int_K |f - h|\, d\mathcal{L}^n \\ &\qquad \le ||\partial[g]||(K) + ||\partial[f \vee y - y]||(K) + \lambda \int_K |f - h|\, d\mathcal{L}^n \\ &\qquad \le ||\partial[g]||(K) + ||\partial[f \vee y]||(K) + \lambda \int_K |f \wedge y - g|\, d\mathcal{L}^n\end{aligned}$$

Thus $f \wedge y \in \mathcal{B}_\lambda(\Omega)$.

Finally,

$$f \vee y = -\left((-f) \wedge (-y)\right) \in \mathcal{B}_\lambda(\Omega).$$

□

Lemma 5.2.1. *Suppose* $\lambda \in [0, \infty)$ *and* $f \in \mathcal{B}_\lambda(\Omega)$, $g \in \mathbf{BV}^{loc}(\Omega)$,

$$\mathbf{ess\ inf} f \le g \le \mathbf{ess\ sup} f,$$

K is a compact subset of Ω, $u(x) = \mathbf{dist}\,(x, K)$ for $x \in \Omega$, $0 < h < \infty$ and $\{u \le h\}$ is a compact subset of Ω. Then

$$||\partial[f]||(K) \le ||\partial[g]||(\{u \le h\}) + \left(\lambda + \frac{1}{h}\right) \int_{\{u \le h\}} |f - g|\, d\mathcal{L}^n.$$

In particular, if $\mathbf{ess\ inf} f \le y \le \mathbf{ess\ sup} f$ *then*

$$||\partial[f]||(K) \le \left(\lambda + \frac{1}{h}\right) \int_{\{u \le h\}} |f - y|\, d\mathcal{L}^n.$$

Proof. For each $r \in (0, h)$ let $h_r = g 1_{\{u \le r\}} + f 1_{\{u > r\}}$. Then $\mathbf{spt}\,[f - h_r] \subset \{u \le r\}$, $f - h_r = (f - g) 1_{\{u \le r\}}$ and, by (4.4.5),

$$||\partial[h_r]||(\{u \le r\}) \le || < [g] - [f], u, r > ||(\{u \le r\}) + ||\partial[g]||(\{u \le r\})$$

so

$$\begin{aligned} &||\partial[f]||(\{u \le r\}) \\ &\quad \le ||\partial[h_r]||(\{u \le r\}) + \lambda \int_{\{u \le r\}} |f - h_r| \\ &\quad \le || < [g] - [f], u, r > ||(\{u \le r\}) + ||\partial[g]||(\{u \le r\}) + \lambda \int_{\{u \le r\}} |f - g|\, d\mathcal{L}^n. \end{aligned}$$

Now integrate from 0 to h and make use of (4.4.6) to obtain the first inequality to be proved; to obtain the second set $g(x) = y$ for $x \in \Omega$. □

Theorem 5.2.4. *Suppose $\lambda \in [0, \infty)$, f is a sequence in $\mathcal{B}_\lambda(\Omega)$, $F \in \mathbf{L}_1^{loc}(\Omega)$ and $f_\nu \to F$ in $\mathbf{L}_1^{loc}(\Omega)$. Then $F \in \mathcal{B}_\lambda(\Omega)$ and*

$$||\partial[f_\nu]|| \to ||\partial[F]|| \quad \textit{weakly as } \nu \to \infty.$$

Proof. Let K be a compact subset of Ω, let $u(x) = \mathbf{dist}\,(x, K)$ for $x \in \Omega$ and let R be the supremum of those $r \in (0, \infty)$ such that $\{u \le r\}$ is a compact subset of Ω.

Suppose $F = 0$. Let $h \in (0, R)$. For each $\nu \in \mathbf{P}$ let y_ν be the average of f_ν on $\{u \le h\}$. Then

$$||\partial[f_\nu]||(K) \le (\lambda + \frac{1}{h}) \int_{\{u \le h\}} |f_\nu - y_\nu|\, d\mathcal{L}^n \to 0 \quad \text{as } \nu \to \infty.$$

Owing to the arbitrariness of K we find that $||\partial[f_\nu]||$ tends weakly to zero so that the Theorem holds in this case.

Suppose $F \ne 0$. Then

$$\limsup_{\nu \to \infty} \mathbf{ess\ inf} f_\nu \le \mathbf{ess\ inf} F < \mathbf{ess\ sup} F \le \liminf_{\nu \to \infty} \mathbf{ess\ sup} f_\nu. \tag{5.2.2}$$

For any open subset U of Ω we have from 4.2.2 that

$$||\partial[F]||(U) \le \liminf_{\nu \to \infty} ||\partial[f_\nu]||(U). \tag{5.2.3}$$

For each m, M such that

$$\mathbf{ess\ inf} F < m \le M < \mathbf{ess\ sup} F$$

we choose a $N(m, M) \in \mathbf{P}$ such that

$$\mathbf{ess\ inf} f_\nu \le m \quad \text{and} \quad M \le \mathbf{ess\ sup} f_\nu \quad \text{provided } \nu \ge N(m, M); \tag{5.2.4}$$

it follows that

$$\mathbf{ess\ inf}\, f_\nu \le F_{m,M} \le \mathbf{ess\ sup}\, f_\nu \quad \text{whenever } \nu \ge N(m,M).$$

where we have set $F_{m,M} = (F \wedge M) \vee m$. For any $r \in (0,R)$ we infer from (5.2.4) and 5.2.1 that

$$\begin{aligned} ||\partial[f_\nu]||(K) &\le ||\partial[F_{m,M}||(\{u \le r\}) + \left(\lambda + \frac{1}{h}\right) \int_{\{u \le h\}} |f_\nu - F_{m,M}|\, d\mathcal{L}^n \\ &\to ||\partial[F_{m,M}||(\{u \le r\}) + \left(\lambda + \frac{1}{h}\right) \int_{\{u \le h\}} |F - F_{m,M}|. \end{aligned}$$

Letting $m \downarrow \mathbf{ess\ inf}\, F$ and $M \uparrow \mathbf{ess\ sup}\, F$, using (4.9.1) and letting $r \downarrow 0$ we find that

$$\limsup_{\nu \to \infty} ||\partial[f_\nu]||(K) \le ||\partial[F]||(K).$$

In view of (4.2.2) we find that

$$||\partial[f_\nu]|| \to ||\partial[F]|| \quad \text{weakly as } \nu \to \infty. \tag{5.2.5}$$

We now show that $F \in \mathcal{B}_\lambda(\Omega)$. To this end, let $G \in \mathbf{BV}^{loc}(\Omega) \cap \mathbf{c}(F,K)$. Suppose $\mathbf{ess\ inf}\, F < m < M < \mathbf{ess\ inf}\, F$ and let $G_{m,M} = (G \wedge M) \vee m$. For each $\nu \in \mathbf{P}$ not less than $N(m,M)$ and each $\rho \in (0,R)$ let

$$g_{\nu,\rho} = G_{m,M} \llcorner \{u \le \rho\} + f_\nu \llcorner \{u > \rho\}.$$

Since $g_{\nu,\rho} \in \mathbf{c}(f_\nu, \{u \le \rho\})$ and since $f_\nu - g_{\nu,\rho} = (f_\nu - G_{m,M}) 1_{\{u \le \rho\}}$ we infer that

$$||\partial[f_\nu]||(\{u \le \rho\}) \le ||\partial[g_{\nu,\rho}||(\{u \le \rho\}) + \lambda \int_{\{u \le \rho\}} |f_\nu - G_{m,M}|\, d\mathcal{L}^n.$$

Suppose $0 < r < s < R$. Keeping in mind that $G_{m,M} - f_\nu = F_{m,M} - f_\nu$ on $\Omega \sim I$ we use (4.4.6) to obtain

$$\begin{aligned} &\int_r^s ||\partial[g_{\nu,r}]||(\{u \le \rho\})\, d\mathcal{L}^1 \rho \\ &\qquad \le \int_{\{r < u < s\}} |F_{m,M} - f_\nu|\, d\mathcal{L}^n + \int_r^s ||\partial[G_{m,M}||(\{u \le \rho\})\, d\mathcal{L}^1 \rho. \end{aligned}$$

It follows that

$$\begin{aligned} (s-r)||\partial[f_\nu]||(\{u \le r\}) &\le \int_r^s ||\partial[f_\nu]||(\{u \le \rho\})\, d\mathcal{L}^1 \rho \\ &\le \int_{\{r < u < s\}} |F_{m,M} - f_\nu|\, d\mathcal{L}^n + (s-r)||\partial[G_{m,M}||(\{u \le s\}) \\ &\qquad + \lambda(s-r) \int_{\{u \le s\}} |f_\nu - G_{m,M}|\, d\mathcal{L}^n \end{aligned}$$

Letting $m \downarrow \mathbf{ess\ inf}\, F$ and $M \uparrow \mathbf{ess\ sup}\, F$ and then letting $\nu \to \infty$ we find that

$$\limsup_{\nu \to \infty} ||\partial[f_\nu]||(\{u \le r\}) \le ||\partial[G]||(\{u \le s\}) + \lambda \int_{\{u \le s\}} |F - G|\, d\mathcal{L}^n.$$

Owing to the arbitrariness of r, s we infer from (5.2.5) that

$$||\partial[G]||(K) \le ||\partial[F]||(K) + \lambda \int_K |F - G|\, d\mathcal{L}^n,$$

as desired. □

Theorem 5.2.5. *Suppose $\mathcal{E}$ is a nonempty nested subfamily of $\mathcal{C}_\lambda(\Omega)$. Then $\cup\mathcal{E}$ and $\cap\mathcal{E}$ belong the $\mathcal{C}_\lambda(\Omega)$.*

Proof. Let D be a nondecreasing sequence in $\mathcal{E}$ such that

$$\Sigma_K(\cup\mathcal{E}, \cup_{\nu=1}^\infty D_\nu) = 0$$

whenever K is a compact subset of Ω. From Theorem 5.2.1 we infer that $1_{E_\nu} \in \mathcal{B}_\lambda(\Omega)$ for $\nu \in \mathbf{P}$. From the preceding Theorem we infer that $1_{\cup\mathcal{E}} \in \mathcal{B}_\lambda(\Omega)$. From (the trivial part of) Theorem 5.2.1 we infer that $\cup\mathcal{E} \in \mathcal{C}_\lambda(\Omega)$.

To show that $\cap\mathcal{E} \in \mathcal{C}_\lambda(\Omega)$ on chooses a nonincreasing sequence D in $\mathcal{E}$ such that

$$\Sigma_K(\cap\mathcal{E}, \cap_{\nu=1}^\infty D_\nu) = 0$$

and proceeds as in the preceding paragraph. □

Theorem 5.2.6. *Suppose $f \in \mathcal{B}_\lambda(\Omega)$ and $y \in \mathbf{R}$. Then*

$$\{f < y\},\ \{f \le y\},\ \{f > y\},\ \{f \ge y\} \in \mathcal{C}_\lambda(\Omega).$$

Proof. For each $\nu \in \mathbf{P}$ let

$$g_\nu = \nu\left(\left((f-y) \wedge \frac{1}{\nu}\right) \vee 0\right)$$

and note that, in view of the foregoing,

$$g_\nu \in \mathcal{B}_\lambda(\Omega).$$

One readily verifies that

$$g_\nu \uparrow 1_{\{f>y\}} \text{ as } \nu \uparrow \infty$$

so that, by the Theorem 5.2.4,

$$1_{\{f>y\}} \in \mathcal{B}_\lambda(\Omega).$$

Since $\{f \ge y\} = \cap_{z<y}\{f > z\}$ we infer from the preceding Theorem that $\{f \ge y\} \in \mathcal{B}_\lambda(\Omega)$. The remaining statements to be proved now follow from Proposition 5.2.1. □

Theorem 5.2.7. *Suppose $f \in \mathbf{BV}^{loc}(\Omega)$ and D is a dense set in $\mathbb{R}$ such that at least one of*

$$\{f > y\}, \quad \{f \ge y\}, \quad \{f < y\}, \quad \{f \le y\}$$

belongs to $\mathcal{C}_\lambda(\Omega)$ whenever $y \in D$. Then

$$f \in \mathcal{B}_\lambda(\Omega).$$

Proof. Let

$$A = \{z \in \mathbb{R} : \{f > z\} \in \mathcal{C}_\lambda(\Omega)\} \quad \text{and let} \quad B = \{z \in \mathbb{R} : \{f \ge z\} \in \mathcal{C}_\lambda(\Omega)\}.$$

It follows from Proposition 5.2.1 that $A \cup B$ is dense in $\mathbb{R}$. Suppose $y \in \mathbb{R}$. Then

$$\{f \ge y\} = \left(\cap_{z\in A,\ z<y}\{f > z\}\right) \cup \left(\cap_{z\in B,\ z<y}\{f > y\}\right).$$

It follows from Theorem 5.2.5 that $\{f \ge y\} \in \mathcal{C}_\lambda(\Omega)$.

Suppose K is a compact subset of Ω and $g \in \mathbf{BV}^{loc}(\Omega) \cap \mathbf{c}(f, K)$. Keeping mind (4.5.5) we infer from (4.9.1) and (4.10.2) to that

$$\begin{aligned} ||\partial[f]||(K) &= \int_{-\infty}^{\infty} ||\partial[1_{\{f \geq y\}}]||(K)\, d\mathcal{L}^1 y \\ &\leq \int_{-\infty}^{\infty} \left(||\partial[1_{\{g \geq y\}}]||(K) + \lambda \int |1_{\{f \geq y\}} - 1_{\{g \geq y\}}|\, d\mathcal{L}^n \right) d\mathcal{L}^1 y \\ &= ||\partial[g]||(K) + \lambda \int |f - g|\, d\mathcal{L}^n. \end{aligned}$$

□

5.3. **Generalized mean curvature.**

Theorem 5.3.1. *Suppose $\lambda \in [0, \infty)$, $D \in \mathcal{C}_\lambda(\Omega)$ and $X \in \mathcal{X}(\Omega)$. Then*

$$\int \operatorname{trace} P(x) \circ \partial X(x) \circ P(x)\, d||\partial[D]||x \leq \lambda \int |X|\, d||\partial[D]||$$

where, for each $x \in \mathbf{b}(D)$, we have let $P(x)$ be orthogonal projection of $\mathbb{R}^n$ onto $\{v \in \mathbb{R}^n : v \bullet \mathbf{n}_D(x) = 0\}$.

Remark 5.3.1. *We restate this Theorem in the language of* [AW1]*. Let V be the $n-1$ dimensional varifold in Ω naturally associated to $\partial[D]$; that is,*

$$V(B) = \mathcal{H}^{n-1}(\{x \in \mathbf{b}(D) : (x, \{v \in \mathbb{R}^n : v \bullet \mathbf{n}_D(x) = 0\} \in B\})$$

whenever B is a Borel subset of the product of Ω with the Grassmann manifold of $n-1$ dimensional linear subspaces of $\mathbb{R}^n$. Then the preceding Theorem amounts to saying that

$$|\delta V(X)| \leq \lambda \int |X|\, d||V|| \quad \textit{whenever } X \in \mathcal{X}(\Omega).$$

Thus, if δV is as in [AW1, 4.2]*,*

$$||\delta V|| \leq \lambda ||V||.$$

δV could reasonably be called the **generalized mean curvature of** *V.*

Proof. Let $K = \mathbf{spt}\, X$ and let h, I be such that $(I, h, K) \in \mathcal{V}(\Omega)$ and $\dot{h}_0 = X$. For each $t \in I$ let $E_t = \{h_t(x) : x \in D\}$ and let $A(t) = ||\partial[E_t]||(K)$. Then for any *positive* $t \in I$ we infer from Proposition 4.12.1 that

$$\begin{aligned} A(t) - A(0) &\leq \lambda \int_\Omega |1_{E_t} - 1_D|\, d\mathcal{L}^n \\ &= ||[E_t] - [D]||(K) \\ &\leq \int_0^t \left(\int |X| ||\partial \dot{h}_\tau(x)||^{n-1}\, d||\partial[D]||x \right) d\mathcal{L}^1 \tau. \end{aligned}$$

To complete the proof we let $t \downarrow 0$ and invoke Theorem 4.12.1. □

5.4. **Monotonicity.**

Theorem 5.4.1 (The Monotonicity Theorem)**.** *Suppose* $\lambda \in [0, \infty)$ *and* $D \in \mathcal{C}_\lambda(\Omega)$. *Then*

$$e^{\lambda r}\Theta^{n-1}(||\partial[D]||, a, r)$$

is nondecreasing as a function of $r \in (0, \mathbf{dist}\,(a, \mathbb{R}^n \sim \Omega))$ *for each* $a \in \Omega$. *Moreover,*

$$\Theta^{n-1}(||\partial[D]||, a)$$

exists for each $a \in \Omega$ *and depends uppersemicontinuously on* a. *Finally,*

$$\Theta^{n-1}(||\partial[D]||, a) \geq 1 \quad \textit{whenever } a \in \mathbf{spt}\,\partial[D].$$

Proof. In view of Corollary 5.2.1,5.3.1 and (4.8.1)(iii) this follows from [AW1, 5.1]. □

Corollary 5.4.1. *Suppose* $0 \leq \lambda < \infty$, $D \in \mathcal{C}_\lambda(\Omega)$ *and* $a \in \mathbf{spt}\,\partial[D]$. *Then*

$$e^{-\lambda r}\alpha(n-1)r^{n-1} \leq ||\partial[D]||(\mathbf{U}^n(a, r))$$

$$e^{-\lambda r}\frac{\alpha(n-1)}{n}r^n \leq (1 + \lambda r)\mathcal{L}^n(D \cap \mathbf{U}^n(a, r))$$

whenever $0 < r < \mathbf{dist}\,(a, \mathbb{R}^n \sim \Omega)$. *Moreover, if* $\Omega = \mathbb{R}^n$ *and* $\mathcal{L}^n(D) < \infty$ *then* $\mathbf{spt}\,[D]$ *is compact.*

Proof. The first inequality to be proved follows directly from the Monotonicity Theorem.

Suppose $0 < r < \mathbf{dist}\,(a, \mathbb{R}^n \sim \Omega)$. For each $\rho \in (0, r)$ let $E_\rho = D \cap \{u > \rho\}$ where we have set $u(x) = |x - a|$ for $x \in \Omega$. Whenever $0 < \rho < r$ we have

$$||\partial[E_\rho]|| = ||\partial D]|| \llcorner \{u > \rho\} + || < D, u, \rho > ||$$

so that, by the Monotonicity Theorem, (4.4) and (4.10.2),

$$\begin{aligned} e^{-\lambda r}\alpha(n-1)\rho^{n-1} &\leq e^{-\lambda\rho}\alpha(n-1)\rho^{n-1} \\ &\leq ||\partial[D]||(\{u \leq \rho\}) \\ &\leq ||\partial[E_\rho]||(\{u \leq \rho\}) + \lambda\int_\Omega |1_D - 1_{E_\rho}| \\ &= \mathbf{M}(< D, u, \rho >) + \lambda\mathcal{L}^n(D \cap \mathbf{B}^n(a, \rho)). \end{aligned}$$

Now integrate this inequality over $(0, r)$ and make use of (4.4.6). □

Corollary 5.4.2. *Suppose*

$$0 < R < \infty, \quad 0 < r < \infty, \quad a \in \Omega \quad \textit{and} \quad R + r \leq \mathbf{dist}\,(a, \mathbb{R}^n \sim \Omega);$$
$$f \in \mathcal{B}_\lambda(\Omega);$$

and

$$Y = \{y \in \mathbb{R} : ||\partial[\{f \geq y\}]||(\mathbf{U}^n(a, R)) > 0\}.$$

Then

$$\mathcal{L}^1(Y)e^{-\lambda r}\alpha(n-1)r^{n-1} \leq ||\partial[f]||(\mathbf{U}^n(a, R + r))$$

and

$$\mathcal{L}^1(Y)e^{-\lambda r}\frac{\alpha(n-1)}{n}r^n \leq (1 + \lambda r)\int_{\mathbf{U}^n(a, R+r)} |f|\, d\mathcal{L}^n.$$

Proof. For each $y \in Y \sim \{0\}$ we apply the preceding Corollary with D there equal to $\{f \geq y\}$. □

5.5. The Regularity Theorem for $\mathcal{C}_\lambda(\Omega)$.

Definition 5.5.1. *Whenever*

$$a \in \mathbb{R}^n, \quad 0 < r < \infty, \quad 0 < \mu \le 1, \quad 0 \le \beta < \infty$$

we let

$$\mathbf{R}(a, r, \mu, \beta)$$

be the family of closed subsets S of $\mathbb{R}^n$ such that there are

$$\Psi, \quad U, \quad g$$

with the following properties:

(i) *Ψ is an orientation preserving isometry from $\mathbb{R}^n$ onto $\mathbb{R}^{n-1} \times \mathbb{R}$ and $\Psi(a) = (0,0)$;*
(ii) *$g : \mathbf{U}^{n-1}(0,r) \to \mathbf{U}^1(0,r)$, $g(0) = 0$, g is continuously differentiable and $\partial g(0) = 0$;*
(iii) *$|\partial g(u) - \partial g(v)| \le \beta \left(\frac{|u-v|}{r}\right)^\mu$ whenever $u, v \in \mathbf{U}^{n-1}(0,r)$;*
(iv) *$\Psi[S] \cap (\mathbf{U}^{n-1}(0,r) \times \mathbf{U}^1(0,r)) = \{(u,v) \in \mathbf{U}^{n-1}(0,r) \times \mathbf{U}^1(0,r) : v \le g(u)\}$.*

Remark 5.5.1. *If $h \in \mathbb{R}^n$, $0 < \eta < \infty$ and L is a linear isometry of $\mathbb{R}^n$ then*

$$S \in \mathbf{R}(a, r, \mu, \beta) \;\Leftrightarrow\; \{L(\eta(x+h) : x \in S\} \in \mathbf{R}(L(\eta(a+h), \eta r, \mu, \beta).$$

Theorem 5.5.1 (Regularity Theorem for $\mathcal{C}_\lambda(\Omega)$)**.** *Suppose*

$$0 < \mu < 1 \quad \textit{and} \quad 0 < \beta < \infty.$$

There exists $\theta \in (0, \sqrt{2}/2)$ such that if

$$0 \le \lambda < \infty; \quad 0 < R < \infty; \quad \lambda R \le \theta; \quad r = \theta R;$$

$$E \in \mathcal{C}_\lambda(\Omega) \quad S = \mathbf{spt}\,[E];$$

$$a \in \mathbf{bdry}\, S \quad \textit{and} \quad \mathbf{U}^n(a, R) \subset \Omega$$

then

$$[E] = [S] \quad \textit{and} \quad S \in \mathbf{R}(a, r, \mu, \beta).$$

Remark 5.5.2. *In case $n = 2$ the Regularity Theorem also holds with $\mu = 1$.*

The Regularity Theorem will be proved by systematic applying the ideas of the regularity theory in the context of geometric measure theory for surfaces of which nearly minimize area.

In view of 5.2.6 and the Regularity Theorem of [AW1, 8] the present Regularity Theorem 5.5.1 will follow from the following Lemma.

Lemma 5.5.1. *Suppose*

$$1 < \zeta < \infty.$$

There exists $\eta \in (0,1)$ such that if $0 \le \lambda < \infty$; $a \in \mathbb{R}^n$; $0 < R < \infty$;

$$\lambda R \le \eta; \quad E \in \mathcal{C}_\lambda(\mathbf{U}^n(a, R)); \quad \textit{and} \quad a \in \mathbf{spt}\,\partial[E]$$

then

$$\Theta^{n-1}(||\partial[E]||, a, \eta R) \le \zeta.$$

Proof. Owing to the way the various entities in the Lemma change under application of homotheties and translations we find that we may assume without loss of generality that $a = 0$ and $R = 1$.

Suppose the Lemma were false. Then there would exist $\zeta \in (1, \infty)$; a sequence η in $(0, 1)$ with limit zero; and sequences E, λ such that, for each $\nu \in \mathbf{P}$,

$$\lambda_\nu \leq \eta_\nu; \quad E_\nu \in \mathcal{C}_{\lambda_\nu}(\mathbf{U}^n(0,1)) \quad \text{and} \quad 0 \in \mathbf{spt}\, \partial[E_\nu]$$

but such that

$$\Theta^{n-1}(||\partial[E_\nu]||, 0, \eta_\nu) > \zeta. \tag{5.5.1}$$

From the Monotonicity Theorem we have

$$(0,1) \ni t \mapsto e^{\lambda_\nu t}\Theta^{n-1}(||\partial[E_\nu], 0, t) \quad \text{is nondecreasing} \tag{5.5.2}$$

for each $\nu \in \mathbf{P}$.

Replacing E by a subsequence if necessary we may use 4.9.2 and 5.2.4 to obtain a Lebesgue measurable subset F of $\mathbf{U}^n(0,1)$ such that $E_\nu \to F$ in $\mathbf{L}_1^{loc}(\mathbf{U}^n(0,1))$ as $\nu \to \infty$,

$$F \in \bigcap_{\nu=1}^{\infty} \mathcal{C}_{\lambda_\nu}(\mathbf{U}^n(0,1)) = \mathcal{C}_0(\mathbf{U}^n(0,1)) \tag{5.5.3}$$

and

$$||\partial[E_\nu]|| \to ||\partial[F]|| \quad \text{weakly as } \nu \to \infty. \tag{5.5.4}$$

Letting B equal the set of $t \in (0,1)$ such that $||\partial[F]||(\{x \in \mathbb{R}^n : |x| = t\})$ is positive we observed that B is countable and infer from (4.2.2) that

$$\lim_{\nu\to\infty} \Theta^{n-1}(||\partial[E_\nu], 0, t) = \Theta^{n-1}(||\partial[F], 0, t) \quad \text{for any } t \in (0,1) \sim B.$$

This together with (5.5.1), (5.4.1) and the fact that $\lambda_\nu \to 0$ as $\nu \to \infty$ implies

$$\Theta^{n-1}(||\partial[F]||, 0, t) \geq \zeta \quad \text{whenever } t \in (0,1) \sim B. \tag{5.5.5}$$

As $F \in \mathcal{C}_0(\mathbf{U}^n(0,1))$ we find that $\partial[F]$ is an absolutely area minimizing integral current of dimension $n-1$ in $\mathbf{U}^n(0,1)$. As 4.8.1 implies that

$$\Theta^{n-1}(||\partial[F]||, x) = 1 \quad \text{for } ||\partial[F]|| \text{ almost all } x$$

it follows from the Regularity Theorem of [FE, 5.4.15] that $\partial[F]$ is integration over an oriented $n-1$ dimensional real analytic hypersurface M of $\mathbf{U}^n(0,1)$. Consequently, $\Theta^{n-1}(||\partial[F]||, 0) = 1$ which is incompatible with (5.5.5). □

Remark 5.5.3. *Suppose $n = 2$. Then one can do a little better than the preceding Theorem as follows.*

Let

$$\mathbf{w}(m) = \sqrt{1 + m^2} \quad \textit{for } m \in \mathbf{R}.$$

Suppose I, J are nonempty open intervals, $g : I \to J$ is continuously differentiable, $0 \leq \lambda < \infty$ and

$$E = \{(u,v) \in I \times J : v \leq g(u)\} \in \mathcal{C}_\lambda(I \times J).$$

Then

$$\mathbf{Lip}(\mathbf{w}' \circ g') \leq \lambda. \tag{5.5.6}$$

Note that if g is twice differentiable at $t \in I$ then

$$\left(\mathbf{w}' \circ g'\right)'(t) = \frac{g''}{\mathbf{w}(g')^{3/2}}(t).$$

We prove this as follows. Let $\phi \in \mathcal{D}(I)$ and let $(G, h, K) \in \mathcal{V}(I \times J)$ be such that $h_t(u, v) = (u, v + t\phi(u))$ whenever $(t, (u, v)) \in G \times (I \times J)$. Then

$$\frac{d}{dt} ||h_{t\#}\partial[E]||(I \times J)\big|_{t=0} = \int_I \frac{d}{dt}\mathbf{w}(g' + t\phi')\big|_{t=0}\, d\mathcal{L}^1 = \int_I \frac{g'\phi'}{\mathbf{w} \circ g'}\, d\mathcal{L}^1.$$

Moreover,

$$||h_{t\#}[E] - [E]||(I \times J) = |t| \int_I |\phi|\, d\mathcal{L}^1.$$

To obtain (5.5.6) we let ϕ approximate the indicator function of an compact subinterval of I.

6. Admissibility.

Ultimately we have in mind the study of $F \in \mathbf{F}\Omega$ as in 1.3. For the time being, it will be more convenient to consider a wider class F's.

We will prove theorems about the regularity and geometry of the reduced boundary (see (4.8) of $\{f \geq y\}$, $y \in \mathbb{R}$, where $f \in \mathbf{m}_\epsilon(F)$ and where F satisfies certain conditions which we describe below. These conditions will be satisfied for large classes of functionals which arise in the denoising literature including those described in the Introduction.

All of our theorems will require that $F \in \mathbf{F}(\Omega)$ and $M \in \mathbf{M}(\Omega)$ be *admissible*, a notion we now define.

Definition 6.0.2. *Suppose $F \in \mathbf{F}(\Omega)$. For each $Y \in [0, \infty)$ we let*

$$\mathbf{l}(F, Y)$$

be the infimum of the set of $L \in [0, \infty]$ such that

$$|F(f) - F(g)| \leq L \int_\Omega |f - g|\, d\mathcal{L}^n$$

whenever $f, g \in \mathcal{F}(\Omega)$ and $\mathbf{ess\ sup}\, |f| \vee |g| \leq Y$. We say F is **admissible** *if $\mathbf{l}(F, Y) < \infty$ whenever $Y \in [0, \infty)$.*

Remark 6.0.4. *Suppose $F \in \mathbf{F}(\Omega)$, F is admissible and $\kappa : \mathbb{R} \to \mathbb{R}$ is locally Lipschitzian. It follows that $\kappa \circ F$ is admissible.*

Remark 6.0.5. *Suppose $F \in \mathbf{F}(\Omega)$, F is admissible, $\Lambda : \mathcal{F}(\Omega) \to \mathcal{F}(\Omega)$ and, for each $Z \in [0, \infty)$, there is $M \in [0, \infty)$ such that*

$$\int_\Omega |\Lambda(f) - \Lambda(g)| \leq M \int_\Omega |f - g|\, d\mathcal{L}^n$$

whenever

$$f, g \in \mathcal{F}(\Omega) \quad \text{and} \quad \mathbf{ess\ sup}\, |f| \vee |g| \leq Z.$$

then $F \circ \Lambda$ is admissible. For example, if $\Omega = \mathbb{R}^n$, $k \in \mathbf{L}_1(\mathbb{R}^n)$ and $\Lambda(f) = k * f$ then the above condition holds with $M = \int |k|\, d\mathcal{L}^n$ for any $Z \in [0, \infty)$.

The following simple Proposition relates the notion of admissibility to the spaces $\mathcal{B}_\lambda(\Omega)$.

Proposition 6.0.1. *Suppose* $0 < \epsilon < \infty$*,* F *is admissible,* $f \in \mathbf{m}_\epsilon(F)$ *and*

$$\lambda = \frac{\mathbf{l}(F, \mathbf{ess\ sup}\,|f|)}{\epsilon}.$$

Then $f \in \mathcal{B}_\lambda(\Omega)$*.*

Proof. Suppose $g \in \mathbf{c}(f, K)$. Then

$$\epsilon\left(||\partial[f]||(K) - ||\partial[g]||(K)\right) \le F(f) - F(g) \le \mathbf{l}(F, \mathbf{ess\ sup}\,|f|) \int_\Omega |f - g|.$$

□

Thus the Regularity Theorem 5.5.1 for $\mathcal{C}_\lambda(\Omega)$ applies to the sets $\{f \ge y\}$, $y \in \mathbb{R}$.

Definition 6.0.3. *Suppose* $M \in \mathbf{M}(\Omega)$*. We let*

$$\mathbf{l}(M)$$

be the infimum of the set of nonnegative real numbers L *such that*

$$|M(D) - M(E)| \le L\,\Sigma_\Omega(D, E) \quad \textit{whenever } D, E \in \mathcal{M}(\Omega).$$

We say M *is* **admissible** *if* $\mathbf{l}(M) < \infty$*.*

The following even simpler Proposition is analogous to the preceding Proposition.

Proposition 6.0.2. *Suppose* $0 < \epsilon < \infty$*,* $M \in \mathbf{M}(\Omega)$*,* M *is admissible,* $D \in \mathbf{n}_\epsilon(F)$ *and*

$$\lambda = \frac{\mathbf{l}(M)}{\epsilon}.$$

Then $D \in \mathcal{C}_\lambda(\Omega)$*.*

Proof. Proceed as in the proof of the preceding Proposition. □

Thus the Regularity Theorem 5.5.1 for $\mathcal{C}_\lambda(\Omega)$ applies to the set D.

6.1. **The functionals** N_S**.** The simplest and perhaps the most useful admissible members of $\mathbf{M}(\Omega)$ are defined as follows.

Definition 6.1.1. *Suppose* $S \in \mathcal{M}(\Omega)$*. We define*

$$N_S \in \mathbf{M}(\Omega)$$

by setting

$$N_S(E) = \Sigma_\Omega(S, E) \quad \textit{whenever } E \in \mathcal{M}(\Omega).$$

Evidently, $\mathbf{l}(N_S) = 1$ so N_S is admissible.

6.2. **The denoising case, I.** For example, suppose

$$s \in \mathcal{F}(\Omega), \quad \gamma : \mathbb{R} \to \mathbb{R}, \quad \gamma \text{ is locally Lipschitzian}$$

and

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^n x \quad \text{for } f \in \mathcal{F}(\Omega).$$

(For the time being we do *not* assume γ is convex as we did in the Introduction.)

Suppose $f, g \in \mathcal{F}(\Omega)$; obviously,

$$|F(f) - F(g)| \le \int_\Omega |\gamma(f(x) - s(x)) - \gamma(g(x) - s(x))|\, d\mathcal{L}^n x \le L \int_\Omega |f - g|\, d\mathcal{L}^n$$

where L is the Lipschitz constant of γ on the smallest interval containing $\mathbf{ess\ inf}\,(f-s), \mathbf{ess\ inf}\,(g-s), \mathbf{ess\ sup}\,(f-s), \mathbf{ess\ sup}\,(g-s)$. Thus if $Y \in [0, \infty)$ then $\mathbf{l}(F, Y)$ is the Lipschitz constant of γ on $[-Y - \mathbf{ess\ sup}\, s, Y - \mathbf{ess\ inf}\, s]$. In particular, F is admissible. Moreover, if $f \in \mathcal{F}(\Omega)$, K is a compact subset of Ω and $g \in \mathbf{c}(f, K)$ we find that L equals the Lipschitz constant of γ on $[\mathbf{ess\ inf}\,(f-s), \mathbf{ess\ sup}\,(f-s)]$. Arguing as in the proof of Proposition 6.0.1 we find that if $0 < \epsilon < \infty$ and $f \in \mathbf{m}_\epsilon(F)$ then $f \in \mathcal{B}_{L/\epsilon}(\Omega)$. So Theorem 1.4.2 now follows from Theorem 5.2.6.

The following Theorem is a direct corollary of Proposition 9.1.3 which will be proved using an elementary calibration argument.

Theorem 6.2.1. *Suppose f and γ are smooth, γ is convex and the gradient of f never vanishes. Then $f \in \mathbf{m}_\epsilon(F)$ if and only if*

$$\operatorname{div} N(x) = -\frac{\gamma'(f(x) - s(x))}{\epsilon} \quad \text{for } \mathcal{L}^n \text{ almost all } x \in \Omega \tag{6.2.1}$$

where we have set $N(x) = |\nabla f(x)|^{-1} \nabla f(x)$ for $x \in \Omega$.

Note that if the hypotheses of the preceding Theorem hold, I is an open interval on which $\gamma'' > 0$ and U is an open subset of Ω such that $f(x) - s(x) \in I$ for $\mathcal{L}^n$ almost all $x \in U$ then s is essentially smooth on U. Of course in denoising applications one wishes to allow s to be highly irregular.

Suppose $f \in \mathbf{m}_\epsilon(F)$. Then, as will be no surprise to one who is familiar with functions of least gradient, f may have essential discontinuities as we shall see in 10. Nonetheless, for *any* $y \in \mathbb{R}$ the reduced boundaries of the sets $\{f \geq y\}$ and $\{f > y\}$ always have have the regularity implied by Theorem 1.4.2 and Theorem 1.4.1.

7. Locality.

If we impose more conditions on $F \in \mathbf{F}(\Omega)$ we will be able to get more detailed information about minimizers. We formulate these conditions in the next two subsections.

7.1. Locality defined.

Definition 7.1.1. *Suppose $M \in \mathbf{M}(\Omega)$. We say M is* **local** *if M is admissible and*

$$\hat{M}(D \cup E) = \hat{M}(D) + \hat{M}(E) \quad \text{whenever } D, E \in \mathcal{M}(\Omega) \text{ and } D \cap E = \emptyset;$$

here we have set

$$\hat{M}(E) = M(E) - M(\emptyset) \quad \text{for } E \in \mathcal{M}(()\Omega).$$

Definition 7.1.2. *Let*

$$\mathcal{S}(\Omega) = \{s \in \mathcal{F}(\Omega) : \mathbf{rng}\, s \text{ is finite}\}.$$

Note that $\mathcal{S}(\Omega)$ is a linear subspace of $\mathcal{F}(\Omega)$ which is dense in $\mathbf{L}_1(\Omega)$. The following Proposition is elementary.

Proposition 7.1.1. *Suppose $M \in \mathbf{M}(\Omega)$ and M is local. Then there is one and only one*

$$J \in \mathbf{F}(\Omega)$$

such that J is admissible and

$$J(y 1_E) = y \hat{M}(E) \quad \text{whenever } y \in \mathbb{R} \text{ and } E \in \mathcal{M}(\Omega).$$

Moreover, J is linear and

$$|J(f) - J(g)| \le \mathbf{l}(M) \int_\Omega |f - g| \, d\mathcal{L}^n \quad \textit{whenever } f, g \in \mathcal{F}(\Omega).$$

Proof. For each $s \in \mathcal{S}(\Omega)$ let

$$j(s) = \sum_{y \in \mathbf{rng}\, s \sim \{0\}} y \hat{M}(\{s = y\}) \in \mathbb{R}.$$

Evidently, $j(cs) = cj(s)$ whenever $c \in \mathbb{R}$ and $s \in \mathcal{S}(\Omega)$. Suppose $s, t \in \mathcal{S}(\Omega)$. For each $(y, w) \in \mathbb{R}^2$ let $C_{y,w} = \{s = y\} \cap \{t = w\}$. Then $\{s + t = z\}$ is the union of the finite disjointed family $\{C_{y,w} : y + w = z\}$ for any $z \in \mathbb{R}$. Moreover, for any $y \in \mathbb{R}$, $\{s = y\}$ is the union of the finite disjointed family $\{C_{y,w} : w \in \mathbb{R}\}$ and, for any $w \in \mathbb{R}$, $\{t = w\}$ is the union of the finite disjointed family $\{C_{y,w} : y \in \mathbb{R}\}$. That $j(s + t) = j(s) + j(t)$ follows easily.

Since the closure of $\mathcal{S}(\Omega)$ in $\mathbf{L}_1(\Omega)$ equals $\mathbf{L}_1(\Omega)$ there is one and only one extension J of j to $\mathcal{F}(\Omega)$ such that $|J(f)| \le \mathbf{l}(M) \int_\Omega |f| \, d\mathcal{L}^n$ whenever $f \in \mathcal{F}(\Omega)$. □

The following Proposition will follow from elementary real analysis and differentiation theory.

Proposition 7.1.2. *If $M \in \mathbf{M}(\Omega)$, M is local and*

$$m(x) = \limsup_{r \downarrow 0} \frac{\hat{M}(\mathbf{B}^n(x, r))}{\mathcal{L}^n(\mathbf{B}^n(x, r))} \quad \textit{for } x \in \Omega \tag{7.1.1}$$

then m is a Borel function, $\sup |m| \le \mathbf{l}(M)$ *and*

$$M(E) = M(\phi) + \int_E m \, d\mathcal{L}^n \quad \textit{whenever } E \in \mathcal{M}(\Omega). \tag{7.1.2}$$

Conversely, if $N \in \mathbf{M}(\Omega)$ and there are $c \in \mathbb{R}$ and $\mu \in \mathbf{L}_\infty(\Omega)$ such that

$$N(E) = c + \int_E \mu \, d\mathcal{L}^n \quad \textit{for } E \in \mathcal{M}(\Omega)$$

then N is local.

Proof. Let G be the set of $(x, r) \in \Omega \times (0, \infty)$ such that $\mathbf{dist}\,(x, \mathbb{R}^n \sim \Omega) < r$. Since M is admissible we find that $G \ni (x, r) \mapsto \hat{M}(\mathbf{B}^n(x, r))$ is locally Lipschitzian from which it follows that m is a Borel function.

Let J be as in the preceding Proposition. Then J is a Daniell integral on $\mathcal{S}(\Omega)$ and the theory of [FE, 2.5,2.9] implies that (7.1.1) holds.

The final assertion of the Proposition is obvious. □

Definition 7.1.3. *Suppose $F \in \mathbf{F}(\Omega)$. We say F is* **local** *if F is admissible and*

$$\hat{F}(f + g) = \hat{F}(f) + \hat{F}(g) \quad \textit{whenever } f, g \in \mathcal{F}(\Omega) \textit{ and } fg = 0$$

here we have set

$$\hat{F}(f) = F(f) - F(0) \quad \textit{for } f \in \mathcal{F}(\Omega).$$

For example, if F is as in 1.3, then F is local owing to the fact that

$$\hat{F}(f) = \int_{\{f \neq 0\}} \gamma(f(x) - s(x)) - \gamma(-s(x)) \, d\mathcal{L}^n x \quad \text{whenever } f \in \mathcal{F}(\Omega).$$

On the other hand, $\kappa \circ F$ as in Remark 6.0.4 is not local unless κ is affine and $F \circ \Lambda$ as in Remark 6.0.5 is not local unless $\int |k|\, d\mathcal{L}^n = 0$ in which case F is constant.

For the remainder of this subsection we assume that $F \in \mathbf{F}(\Omega)$ and that F is local.

It follows immediately from the definitions that

$$|\hat{F}(0)| = 0, \tag{7.1.3}$$

$$|\hat{F}(y1_E)| \le \mathbf{l}(F,Y)|y|\mathcal{L}^n(E), \tag{7.1.4}$$

$$|\hat{F}(y1_E) - \hat{F}(z1_E)| \le \mathbf{l}(F,Y)\mathcal{L}^n(E)|y-z| \tag{7.1.5}$$

whenever $E \in \mathcal{M}(\Omega)$, $|y| \vee |z| \le Y < \infty$. Moreover, for any $y \in \mathbb{R}$,

(7.1.6)

$$|F(y1_D) - F(y1_E)| \le \mathbf{l}(F,|y|)|y| \int_\Omega |1_D - 1_E|\, d\mathcal{L}^n \quad \text{whenever } D, E \in \mathcal{M}(\Omega);$$

Definition 7.1.4. *For each $(x,y) \in \Omega \times \mathbb{R}$ we let*

$$k(x,y) = \limsup_{r \downarrow 0} \frac{\hat{F}(y1_{\mathbf{B}^n(x,r)})}{\mathcal{L}^n(\mathbf{B}^n(x,r))}.$$

For each $y \in \mathbb{R}$ we let

$$k_y(x) = k(x,y) \quad \textit{for } x \in \Omega.$$

Theorem 7.1.1. *We have*

(i) *k is a Borel function;*
(ii) *$k(x,0) = 0$ for $x \in \Omega$;*
(iii) *$|k(x,y) - k(y,z)| \le \mathbf{l}(F,Y)|y-z|$ whenever $x \in \Omega$, $y,z \in \mathbb{R}$ and $|y| \vee |z| \le Y < \infty$;*
(iv)
$$F(f) = F(0) + \int_\Omega k(x,f(x))\, d\mathcal{L}^n x \quad \textit{for } f \in \mathcal{F}(\Omega).$$

Moreover, if $c \in \mathbb{R}$; κ is an $\mathcal{L}^n \times \mathcal{L}^1$ measurable function on $\Omega \times \mathbb{R}$ such that $\kappa(x,0) = 0$ for $\mathcal{L}^n$ almost all $x \in \Omega$; for each $Y \in [0,\infty)$ there is $L \in [0,\infty)$ such that, for $\mathcal{L}^n$ almost all $x \in \Omega$,

$$|\kappa(x,y) - \kappa(x,z)| \le L|y-z| \quad \textit{if } y,z \in \mathbb{R} \textit{ and } |y| \vee |z| \le Y;$$

and $G \in \mathbf{F}(\Omega)$ is such that

$$G(f) = c + \int_\Omega \kappa(x,f(x))\, d\mathcal{L}^n x \quad \textit{for } f \in \mathcal{F}(\Omega)$$

then G is admissible and local.

Proof. The estimates (ii) and (iii) follow directly from (7.1.3) and (7.1.5). Since

$$\Omega \times \mathbb{R} \times (0,\infty) \ni (x,y,r) \mapsto \frac{\hat{F}(y1_{\mathbf{B}^n(x,r)})}{\mathcal{L}^n(\mathbf{B}^n(x,r))}$$

is locally Lipschitzian by virtue of (7.1.5) we find that (i) holds.

Let

$$H(f) = F(0) + \int_\Omega k(x,f(x))\, d\mathcal{L}^n x \quad \text{for } f \in \mathcal{F}(\Omega).$$

The estimate (7.1.5) and the locality of F imply that

$$\mathcal{M}(E) \ni E \mapsto \hat{F}(y1_E)$$

is local. From Theorem 7.1.2 we infer that

$$\hat{F}(E) = \int_E k_y \, d\mathcal{L}^n \quad \text{whenever } y \in \mathbb{R} \text{ and } E \in \mathcal{F}(\Omega).$$

It follows that $F(y1_E) = H(y1_E)$ whenever $y \in \mathbb{R}$ and $E \in \mathcal{M}(\Omega)$. Since F and H are local we find that $F(s) = H(s)$ for $s \in \mathcal{S}(\Omega)$. Since F and H are admissible and $\mathcal{S}(\Omega)$ is dense in $\mathbf{L}_1(\Omega)$ we find that $F = H$.

The final assertion of the Proposition is obvious. □

7.2. **A generalization of the "layer cake" formula.**

Definition 7.2.1. *Whenever $y, z \in \mathbb{R}$ and $y \neq z$ we let*

$$J_{y,z}(E) = \frac{F(z1_E) - F(y1_E)}{z - y} \quad \textit{whenever } E \in \mathcal{M}(\Omega).$$

Proposition 7.2.1. *Suppose $y, z \in \mathbb{R}$ and $y \neq z$. Then $J_{y,z}$ is local. Moreover,*

$$|J_{y,z}(E)| \leq \mathbf{l}(F, Y)\mathcal{L}^n(E) \tag{7.2.1}$$

and

$$|J_{y,z}(D) - J_{y,z}(E)| \leq \mathbf{l}(F, Y)\Sigma_\Omega(D, E) \tag{7.2.2}$$

whenever $|y| \vee |z| \leq Y < \infty$ and $D, E \in \mathcal{M}(\Omega)$.

Proof. The locality of $J_{y,z}$ is a direct consequence of the locality of F. (7.2.1) is an immediate consequence of (7.1.5).

Since F is local we have

$$\begin{aligned}
&F(z1_D) - F(y1_D) - (F(z1_E) - F(y1_E)) \\
&\quad = F(z1_{D\cap E}) + F(z1_{D\sim E}) - (F(y1_{D\cap E}) + F(y1_{D\sim E})) \\
&\qquad - \big((F(z1_{E\cap D}) + F(z1_{E\sim D}) - (F(y1_{E\cap D}) - F(y1_{E\sim D}))\big) \\
&\quad = F(z1_{D\sim E}) - F(y1_{D\sim E}) - (F(z1_{E\sim D}) - F(y1_{E\sim D}));
\end{aligned}$$

(7.2.2) now follows from (7.1.5). □

Definition 7.2.2. *Suppose $y \in \mathbb{R}$. Keeping in mind (7.2.1) we define*

$$L_y, U_y \in \mathbf{M}(\Omega)$$

by letting

$$L_y(E) = \liminf_{z\to y} J_{y,z}(E) \quad \textit{and} \quad U_y(E) = \limsup_{z\to y} J_{y,z}(E)$$

for $E \in \mathcal{M}(\Omega)$

For each $(x, y) \in \Omega \times \mathbb{R}$ we let

$$l(x, y) = \liminf_{r\downarrow 0} \frac{L_y(\mathbf{B}^n(x, r))}{\mathcal{L}^n(\mathbf{B}^n(x, r))} \quad \textit{and we let} \quad u(x, y) = \limsup_{r\downarrow 0} \frac{U_y(\mathbf{B}^n(x, r))}{\mathcal{L}^n(\mathbf{B}^n(x, r))}.$$

For each $y \in \mathbb{R}$ we let $l_y(x) = l(x, y)$ and $u_y(x) = u(x, y)$ whenever $x \in \Omega$.

Theorem 7.2.1. *We have*

(i) *l and u are Borel functions.*

(ii) *$\mathbf{l}(L_y) \le \mathbf{l}(F, Y)$ and $\mathbf{l}(U_y) \le \mathbf{l}(F, Y)$ whenever $y \in \mathbb{R}$ and $|y| < Y < \infty$; in particular L_y and U_y are admissible for any $y \in \mathbb{R}$.*
(iii) *L_y and U_y are local and equal for $\mathcal{L}^1$ almost all y.*
(iv) *For any $f \in \mathcal{F}(\Omega)$ we have that*

$$(-\infty, 0) \ni \mapsto U_y(\{f < y\}) \quad \textit{and} \quad (0, \infty) \ni y \mapsto U_y(\{f \ge y\})$$

are $\mathcal{L}^1$ summable.
(v) *For any $f \in \mathcal{F}(\Omega)$ we have*

$$F(f) = F(0) - \int_{-\infty}^{0} U_y(\{f < y\})\, d\mathcal{L}^1 y + \int_{0}^{\infty} U_y(\{f \ge y\}\, d\mathcal{L}^1 y.$$

Proof. Since

$$\Omega \times \mathbb{R} \times \mathbb{R} \times (0, \infty) \ni (x, y, z, r) \mapsto \frac{J_{y,z}(\mathbf{B}^n(x, r))}{\mathcal{L}^n(\mathbf{B}^n(x, r))}$$

is locally Lipschitzian by virtue of (7.1.5) and (7.1.6) we deduce that l and u are Borel functions.

(ii) follows immediately from (7.2.2).

For each $E \in \mathcal{M}(\Omega)$ let

$$Z(E)$$

be the set of $y \in \mathbb{R}$ such that $\lim_{z \to y} J_{y,z}(E)$ exists and note that $\mathcal{L}^1(\mathbb{R} \sim Z(E)) = 0$ since $\mathbb{R} \ni y \mapsto F(y 1_E)$ is locally Lipschitzian.

Let $\mathcal{D}$ be a countable subfamily of $\mathcal{M}(\Omega)$ which is dense with respect to $\Sigma_\Omega(\cdot, \cdot)$. Let $W = \cap_{D \in \mathcal{D}} Z(D)$ and note that

$$\mathcal{L}^1(\mathbb{R} \sim W) = 0.$$

Suppose $y \in W$. Let $E \in \mathcal{M}(\Omega)$ and let $\eta > 0$. Choose Y such that $|y| < Y < \infty$. Choose $D \in \mathcal{D}$ such that $\mathbf{l}(F, Y)\Sigma_\Omega(D, E) < \eta/3$. Choose $\delta > 0$ such that

$$0 < |w - y| < \delta \text{ and } 0 < |z - y| < \delta \;\Rightarrow |J_{y,w}(D) - J_{y,z}(D)| < \eta/3.$$

Then if $0 < |w - y| < \delta$ and $0 < |z - y| < \delta$ we infer with the help of (7.2.2) that

$$\begin{aligned} |J_{y,w}(E) - J_{y,z}(E)| &\le |J_{y,w}(E) - J_{y,w}(D)| + |J_{y,w}(D) - J_{y,z}(D)| \\ &\quad + |J_{y,z}(D) - J_{y,z}(D)| \\ &< \mathbf{l}(F, Y)\Sigma_\Omega(D, E) + \eta/3 + \mathbf{l}(F, Y)\Sigma_\Omega(D, E) \\ &\le \eta. \end{aligned}$$

It follows that $y \in Z(E)$. Thus $L_y = U_y$. Since $J_{y,z}$ is bounded for $z \in \mathbb{R} \sim \{y\}$ we infer that L_y is local. Thus (iii) holds.

Suppose $f \in \mathcal{F}(\Omega)$ and $\textbf{ess sup}\,|f| \le Y < \infty$. Whenever $y\mathbb{R} \sim \{0\}$ it follows from (7.2.1) that

$$|U_y(\{f \ge y\})| \le \mathbf{l}(F, Y)\mathcal{L}^n(\{f \ge y\}) \quad \text{if } y > 0$$

and

$$|U_y(\{f < y\})| \le \mathbf{l}(F, Y)\mathcal{L}^n(\{f < y\}) \quad \text{if } y < 0$$

so (iv) holds.

It remains to prove (v). For each $f \in \mathcal{F}(\Omega)$ let

$$G(f) = F(0) - \int_{-\infty}^{0} U_y(\{f < y\})\, d\mathcal{L}^1 y + \int_{0}^{\infty} U_y(\{f \ge y\}\, d\mathcal{L}^1 y.$$

Suppose $E \in \mathcal{M}(\Omega)$. Since $\mathbb{R} \ni z \mapsto F(z1_E)$ is locally Lipschitzian we find that $F(y1_E) = G(y1_E)$ for any $y \in \mathbb{R}$. It follows from the locality of F and U_y for $\mathcal{L}^1$ almost all y that F and G agree on $\mathcal{S}(\Omega)$. It then follows from the admissibility of F, the estimates (7.2.1), (7.2.2) and the fact that the closure of $\mathcal{S}(\Omega)$ in $\mathbf{L}_1(\Omega)$ equals $\mathbf{L}_1(\Omega)$ that $F = G$. □

Corollary 7.2.1. *We have $l(x,y) = u(x,y)$ for $\mathcal{L}^n \times \mathcal{L}^1$ almost all $(x,y) \in \Omega \times \mathbb{R}$. Moreover,*

$$k(x,y) = \begin{cases} -\int_y^0 u(x,z)\, d\mathcal{L}^1 z & \text{if } y < 0, \\ \int_0^y u(x,z)\, d\mathcal{L}^1 z & \text{if } y > 0 \end{cases} \tag{7.2.3}$$

whenever $y \in \mathbb{R} \sim \{0\}$.

Proof. If $y \in (0,\infty)$ and $E \in \mathcal{M}(E)$ then, by combining Theorem 7.1.1 (iv), Theorem 7.1.2 and the preceding Theorem one obtains

$$\int_E k(x,y)\, d\mathcal{L}^n x = \hat{F}(y1_E) = \int_0^y U_y(E)\, d\mathcal{L}^n = \int_0^y \left(\int_E l_z\, d\mathcal{L}^n \right) d\mathcal{L}^1 z.$$

One may obtain a similar formula if $y < 0$. □

Theorem 7.2.2. *Suppose $0 < \epsilon < \infty$, $f \in \mathcal{F}(\Omega)$ and, for $\mathcal{L}^1$ almost all $y \in \mathbb{R} \sim \{0\}$*

$$\text{either} \quad y < 0 \text{ and } \{f < y\} \in \mathbf{n}_\epsilon(-U_y) \quad \text{or} \quad y > 0 \text{ and } \{f \geq y\} \in \mathbf{n}_\epsilon(U_y).$$

Then $f \in \mathbf{m}_\epsilon(F)$.

Remark 7.2.1. *Note that for $\mathcal{L}^1$ almost all y we have $[\{f \geq y\}] = [\{f > y\}]$, $[\{f \leq y\}] = [\{f < y\}]$ and $L_y = U_y$.*

Proof. Suppose K is a compact subset of Ω, $g \in \mathcal{F}(\Omega)$ and $\mathbf{spt}\,[f - g] \subset K$. Then for $\mathcal{L}^1$ almost all $y \in (-\infty, 0)$ we have $\Sigma_{\Omega \sim K}(\{f < y\}, \{g < y\}) = 0$ and $\{f < y\} \in \mathbf{n}_\epsilon(-U_y)$ which implies

$$\epsilon ||\partial[\{f < y\}]||(K) - U_y(\{f < y\}) \leq \epsilon ||\partial[\{g < y\}]||(K) - U_y(\{g < y\}).$$

For $\mathcal{L}^1$ almost all $y \in (0,\infty)$ we have $\Sigma_{\Omega \sim K}(\{f \geq y\}, \{g \geq y\}) = 0$ and $\{f \geq y\} \in \mathbf{n}_\epsilon(U_y)$ which implies

$$\epsilon ||\partial[\{f \geq y\}]||(K) + U_y(\{f \geq y\}) \leq \epsilon ||\partial[\{g \geq y\}]||(K) + U_y(\{g \geq y\}).$$

We integrate these inequalities over $(-\infty, 0)$ and $(0,\infty)$, respectively, and use (4.4) and (ii) of Theorem 7.2.1 to complete the proof. □

7.3. **Results when F is convex.**

Theorem 7.3.1. *The following are equivalent.*

(i) *F is convex.*
(ii) *$\mathbb{R} \ni y \mapsto F(y1_E)$ is convex for any $E \in \mathcal{M}(\Omega)$.*
(iii) *For any $x \in \Omega$,*
$$\mathbb{R} \ni y \mapsto k(x,y) \text{ is convex.}$$
(iv) *$\mathbb{R} \ni y \mapsto L_y(E)$ is nondecreasing for any $E \in \mathcal{M}(\Omega)$.*
(v) *$\mathbb{R} \ni y \mapsto U_y(E)$ is nondecreasing for any $E \in \mathcal{M}(\Omega)$.*
(vi) *For any $x \in \Omega$,*
$$\mathbb{R} \ni y \mapsto l(x,y) \text{ is nondecreasing.}$$

(vi) *For any* $x \in \Omega$,

$$\mathbb{R} \ni y \mapsto u(x,y) \text{ is nondecreasing.}$$

Moreover, if F *is convex and* $y \in \mathbb{R}$ *then* L_y *and* U_y *are local,* $L_y = U_y$ *for all but countably many* $y \in \mathbb{R}$,

$$\lim_{z \uparrow y} L_z(E) = L_y(E) \tag{7.3.1}$$

and

$$\lim_{z \downarrow y} U_z(E) = U_y(E) \tag{7.3.2}$$

whenever $E \in \mathcal{F}(\Omega)$.

Proof. That (i) implies (ii) is immediate.

Suppose (ii) holds. Whenever $x \in \Omega$, $0 < r < \infty$, $-\infty < y < z < \infty$ and $0 \le t \le 1$ we have

$$\frac{\hat{F}((1-t)y + tz 1_{\mathbf{B}^n(x,r)})}{\mathcal{L}^n(\mathbf{B}^n(x,r))} \le (1-t)\frac{\hat{F}(y 1_{\mathbf{B}^n(x,r)})}{\mathcal{L}^n(\mathbf{B}^n(x,r))} + t\frac{\hat{F}(z 1_{\mathbf{B}^n(x,r)})}{\mathcal{L}^n(\mathbf{B}^n(x,r))}$$

from which it follows that

$$k(x,(1-t)y + tz) \le (1-t)k(x,y) + tk(x,z).$$

Thus (iii) holds.

If (iii) holds then (i) holds by virtue of Theorem 7.1.1.

Thus (i),(ii) and (iii) are equivalent.

We leave the proof of the following elementary Lemma to the reader.

Lemma 7.3.1. *Suppose* $g : \mathbb{R} \to \mathbb{R}$, g *is absolutely continuous and*

$$h_l(y) = \liminf_{z \to y} \frac{g(z) - g(y)}{z - y} \quad \text{and} \quad h_u(y) = \limsup_{z \to y} \frac{g(z) - g(y)}{z - y}$$

whenever $y \in \mathbb{R}$.

Then g *is convex if and only if* h_l *is nondecreasing if and only if* h_u *is nondecreasing.*

Moreover, if g *is convex then*

$$h_l(y) = \lim_{z \uparrow y} h_l(z) \quad \text{and} \quad h_u(y) = \lim_{z \downarrow y} h_u(z)$$

whenever $y \in \mathbb{R}$.

From the Lemma we infer that (ii) and (iv) are equivalent and that (ii) and (v) are equivalent.

From (iii) of Theorem 7.2.1 we know that L_y and U_y are local for $\mathcal{L}^1$ almost y which implies that (iv) and (vi) are equivalent and that (v) and (vii) are equivalent.

Suppose (iv) holds and $E \in \mathcal{M}(E)$. Since $\mathbb{R} \ni y \mapsto F(y1_E)$ is locally Lipschitzian we have

$$\hat{F}(y1_E) = \begin{cases} \int_0^y L_y(E) d\mathcal{L}^1 z & \text{if } 0 \le y < \infty, \\ -\int_y^0 L_y(E) d\mathcal{L}^1 z & \text{if } -\infty < y < 0. \end{cases}$$

It follows that $\mathbb{R} \ni y \mapsto F(y1_E)$ is convex so (iv) implies (ii). In a similar fashion one may show that (v) implies (ii).

Suppose F is convex. Then (7.3.1) and (7.3.2) hold since $\mathcal{M}(\Omega) \ni E \mapsto \hat{F}(y1_E)$ is convex. Suppose $y \in \mathbb{R}$. By (iii) of Theorem 7.2.1 we may choose an increasing sequence z with limit y such that L_{z_ν} is local for each $\nu \in \mathbf{P}$. From 7.3.1 we infer that

$$L_y(E) = \lim_{\nu \to \infty} L_{z_\nu}(E) \quad \text{whenever } E \in \mathcal{M}(\Omega).$$

It follows that L_y is local. In a similar fashion one may show that U_y is local. Finally, let $\mathcal{C}$ be a countable subset of $\mathcal{M}(\Omega)$ which is dense with respect to $\Sigma_\Omega(\cdot,\cdot)$. Since $\mathcal{M}(\Omega) \ni E \mapsto \hat{F}(y1_E)$ is convex we infer that there is a countable subset C of $\mathbb{R} \sim \{0\}$ such that $L_y|\mathcal{C} = U_y|\mathcal{C}$ for $y \in (\mathbb{R} \sim \{0\}) \sim C$. But if $y \in \mathbb{R} \sim \{0\}$ and $L_y|\mathcal{C} = U_y|\mathcal{C}$ then $L_y = U_y$ since L_y and U_y are admissible. □

Using (iii) of Theorem 7.2.1 and the slicing formula (4.9.1) one easily deduces the following Theorem.

7.4. **Working in the product** $\Omega \times \mathbb{R}$**.** *For the remainder of this section we adopt the notation of 4.11.*

In order to obtain the fundamental Theorems 7.4.2 and 7.4.3 we will use F to define a functional $F^\uparrow$ on subsets of $\Omega \times \mathbb{R}$ which will be very useful in analyzing (ϵ, F)-minimizers. This is one of the main new ideas of the paper. The first of these Theorems is a sort of converse of Theorem 7.2.2. Among other things, it will allow us to obtain the curvature and conjugacy results which follow in 8 and will facilitate the construction of minimizers. Results similar to ours this but in a different context were obtained independently in [AC].

Proposition 7.4.1. *Suppose $B \in \mathcal{G}(\Omega)$. Then*

$$(-\infty, 0) \ni y \mapsto U_y(\{x : (x,y) \in G^-\} \quad \textit{and} \quad (0,\infty) \ni y \mapsto U_y(\{x : (x,y) \in G^+\}$$

are $\mathcal{L}^1$ summable.

Proof. Proceed as in the proof of (iv) of Theorem 7.2.1. □

Definition 7.4.1. *Let*

$$F^\uparrow : \mathcal{G}(\Omega) \to \mathbb{R}$$

be such that

$$F^\uparrow(G) = F(0) - \int_{-\infty}^0 U_y(\{x : (x,y) \in G^-\})\, d\mathcal{L}^1 y + \int_0^\infty U_y(\{x : (x,y) \in G^+\})\, d\mathcal{L}^1 y$$

whenever $G \in \mathcal{G}(\Omega)$.

We have a useful comparison principle.

Theorem 7.4.1. *We have*

$$F(G^\downarrow) \le F^\uparrow(G) \quad \textit{whenever } G \in \mathcal{G}(\Omega).$$

Proof. As we shall see, the Theorem will follow rather directly from the following Lemma.

Lemma 7.4.1. *Suppose $a \in \Omega$, $E \in \mathcal{M}(\mathbb{R})$,*

$$e^+ = \mathcal{L}^1(E \cap (0,\infty)), \quad \textit{and} \quad e^- = \mathcal{L}^1(E \cap (-\infty, 0)).$$

Then

$$k(a, e^+ - e^-) - k(a, 0) \le \left(\int_{E \cap (0,\infty)} - \int_{E \cap (-\infty,0)} \right) u(a,y)\, d\mathcal{L}^1 y.$$

Proof. Suppose $\phi \in \mathcal{D}(\mathbb{R})$ and $0 \le \phi \le 1$. Let

$$J^- = (-\infty, 0), \quad J^+ = (0, \infty), \quad I^\pm = \int_{J^\pm} \phi \, d\mathcal{L}^1$$

and let $\Phi \in \mathcal{E}(\Omega)$ be such that $\Phi' = \phi$ and $\Phi(0) = 0$. Then

$$0 \le \Phi(y) \le y \quad \text{if } y \in J^+ \text{ and } \quad y \le \Phi(y) \le 0 \quad \text{if } y \in J^-. \tag{7.4.1}$$

We let

$$\kappa(y) = k(a, y) \quad \text{whenever } y \in \mathbb{R}.$$

From (7.4.1) and the absolute continuity of κ we infer that

$$\kappa(\Phi(y)) - \kappa(0) = \int_{(0,y)} u(a, \Phi(z))\phi(z) \, d\mathcal{L}^1 z \le \int_{(0,y)} u(a, z)\phi(z) \, d\mathcal{L}^1 z$$

whenever $y \in J^+$ and

$$\kappa(0) - \kappa(\Phi(y)) = \int_{(y,0)} u(a, \Phi(z))\phi(z) \, d\mathcal{L}^1 z \ge \int_{(y,0)} u(a, z)\phi(z) \, d\mathcal{L}^1 z$$

whenever $y \in J^-$. Since $\lim_{y \to \pm\infty} \Phi(y) = \pm I^\pm$ we infer that

$$\kappa(\pm I^\pm) - \kappa(0) \le \pm \int_{J^\pm} u(a, z)\phi(z) \, d\mathcal{L}^1 z.$$

From this inequality and the convexity of κ we find that

$$\begin{aligned}
\kappa(I^+ - I^-) - \kappa(0) &= \kappa(I^+ - I^-) - \kappa(-I^-) + \kappa(-I^-) - \kappa(0) \\
&\le \kappa(I^+) - \kappa(0) + \kappa(-I^-) - \kappa(0) \\
&\le \int_{J^+} u(a, y)\phi \, d\mathcal{L}^1 y - \int_{J^-} u(a, y)\phi \, d\mathcal{L}^1 y
\end{aligned}$$

We let ϕ approximate 1_E to complete the proof of the Lemma. □

From the Lemma we infer that

$$k(x, G^\downarrow(x)) - k(x, 0) \le \int_{\{y:(x,y)\in G^+\}} u(x, y) \, d\mathcal{L}^1 y - \int_{\{y:(x,y)\in G^-\}} u(x, y) \, d\mathcal{L}^1 y$$

for $\mathcal{L}^1$ almost all $x \in \Omega$. Integrating this inequality over Ω we use (iii) of Theorem 7.2.1 and Theorem 7.1.2 to obtain

$$\begin{aligned}
&F(G^\downarrow) - F(0) \\
&\quad \le \int_0^\infty \left(\int_{\{x:(x,y)\in G^+\}} u_y \, d\mathcal{L}^n \right) d\mathcal{L}^1 y - \int_{-\infty}^0 \left(\int_{\{x:(x,y)\in G^-\}} u_y \, d\mathcal{L}^n \right) d\mathcal{L}^1 y \\
&\quad = \int_0^\infty U_y(\{x : (x, y) \in G^+\}) \, d\mathcal{L}^1 y - \int_{-\infty}^0 U_y(\{x : (x, y) \in G^-\}) \, d\mathcal{L}^1 y \\
&\quad = F^\uparrow(G) - F(0),
\end{aligned}$$

as desired. □

Theorem 7.4.2. *Suppose* $0 < \epsilon < \infty$, $f \in \mathbf{m}_\epsilon(F)$ *and* $y \in \mathbb{R} \sim \{0\}$. *Then*

$$\{f > y\} \in \mathbf{n}_\epsilon(U_y) \quad \text{if } y > 0 \quad \text{and} \quad \{f \le y\} \in \mathbf{n}_\epsilon(-U_y) \quad \text{if } y < 0 \tag{7.4.2}$$

and

$$\{f \ge y\} \in \mathbf{n}_\epsilon(L_y) \quad \text{if } y > 0 \quad \text{and} \quad \{f < y\} \in \mathbf{n}_\epsilon(-L_y) \quad \text{if } y < 0. \tag{7.4.3}$$

Proof. We will show that (7.4.2) holds for each $y \in (0, \infty)$ and leave to the reader the straightforward modification of the proof required to show that (7.4.2) holds for $y \in (-\infty, 0)$ and that (7.4.3) holds for each $y \in \mathbb{R} \sim \{0\}$.

For each $y \in \mathbb{R} \sim \{0\}$ we let

$$D_y = \begin{cases} \{f > y\} & \text{if } y > 0, \\ \{f \le y\} & \text{if } y < 0. \end{cases}$$

Suppose $b \in (0, \infty)$, $E \in \mathcal{M}(\Omega)$ and K is a compact subset of Ω such that $\Sigma_{\Omega \sim K}(D_b, E) = 0$. We will show that

$$\epsilon ||\partial[D_b]||(K) + U_b(D_b) \le \epsilon ||\partial[E]||(K) + U_b(E). \tag{7.4.4}$$

Let $v(x) = \mathbf{dist}\,(x, K)$ for $x \in \Omega$ and let R be the supremum of the set of $r \in \mathbb{R}$ such that $\{v \le r\}$ is a compact subset of Ω. Suppose $b < Y < \infty$ and let $Z = (b, Y) \times (0, R)$. For each $(y, r) \in Z$ let

$$C_{y,r} = (E \cap \{v \le r\}) \cup (D_y \cap \{v > r\}) \in \mathcal{M}(\Omega)$$

and note that

$$\Sigma_{\{v > r\}}(D_y, E) \le \mathbf{l}(F, Y)\Sigma_{\Omega \sim K}(D_y, E) = \mathbf{l}(F, Y)\Sigma_{\Omega \sim K}(D_y, D_b). \tag{7.4.5}$$

This implies

$$|U_y(C_{y,r}) - U_y(E)| \le \mathbf{l}(F, Y)\Sigma_{\{v > r\}}(D_y, E) \le \mathbf{l}(F, Y)\Sigma_{\Omega \sim K}(D_y, D_b). \tag{7.4.6}$$

Also, keep in mind that

$$\lim_{y \downarrow b} \Sigma_\Omega(D_y, D_b) = 0. \tag{7.4.7}$$

For each $(y, r) \in Z$ let

$$a(y, r) = \epsilon ||\partial[D_y]||(\{v \le r\}) + U_y(D_y)$$

and let

$$b(y, r) = \epsilon ||\partial[C_{r,y}]||(\{v \le r\}) + U_y(C_{r,y}).$$

Let

$$W = \{(y, r) \in Z : a(y, r) \le b(y, r)\}.$$

Lemma 7.4.2. $\mathcal{L}^2(Z \sim W) = 0$.

Proof. Let $r \in (0, R)$, let I be a bounded open subinterval of (b, Y) and let

$$G = \{(x, y) \in \Omega \times (\mathbb{R} \sim I) : x \in D_y\} \cup \{(x, y) \in \Omega \times I : x \in C_{y,r}\}.$$

Evidently,

$$G^{\downarrow}(x) = f(x) \quad \text{for } \mathcal{L}^n \text{ almost all } x \in \{v > r\}$$

from which it follows that

$$\epsilon ||\partial[f]||(\{v \le r\}) + F(f) \le \epsilon ||\partial[G^{\downarrow}]||(\{v \le r\}) + F(G^{\downarrow}).$$

Let

$$P = \int_{\mathbb{R} \sim I} ||\partial[D_y]||(\{v \le r\})\, d\mathcal{L}^1 y$$

and let

$$Q = -\int_{-\infty}^{0} U)(D_y)\, d\mathcal{L}^1 y + \int_{(0\infty) \sim I} U_y(D_y)\, d\mathcal{L}^1 y.$$

Keep in mind that $||\partial[\{f \geq y\}]|| = ||\partial[D_y]||$ for any $y \in \mathbb{R}$. We have

$$||\partial[f]||(\{v \leq r\}) = P + \int_I ||\partial[D_y||(\{v \leq r\})\, d\mathcal{L}^1 y$$

and

$$F(f) = F(0) + Q + \int_I U_y(D_y)\, d\mathcal{L}^1 y.$$

From Propositions 4.11.3 and 4.11.4 we obtain

$$\begin{aligned} ||\partial[G^{\downarrow}]||(\{v \leq r\}) &\leq ||\partial[G] \llcorner dq||(\{v \leq r\} \times \mathbb{R}) \\ &= \int ||\partial[\{x : (x,y) \in G\}]||(\{v \leq r\})\, d\mathcal{L}^1 y \\ &= P + \int_I ||\partial[C_{y,r}]||(\{v \leq r\}). \end{aligned}$$

From (7.4.1) we obtain

$$F(G^{\downarrow}) \leq F^{\uparrow}(G) = F(0) + Q + \int_I U_y(C_{r,y})\, d\mathcal{L}^1 y.$$

It follows that

$$\int_I a(y,r)\, d\mathcal{L}^1 y \leq \int_I b(y,r)\, d\mathcal{L}^1 y.$$

Owing to the arbitrariness of I find that we infer that

$$\mathcal{L}^1(\{y \in \mathbb{R} \sim \{0\} : (y,r) \notin W\}) = 0$$

so the Lemma is proved. $\square$

Suppose $(y,r) \in Z$. Keeping in mind that $\Sigma_{\Omega \sim K}(E, D_b) = 0$ we infer from (4.4) that

$$\partial[C_{r,y}] \llcorner \{v \leq r\} = \partial[E] \llcorner \{v \leq r\} + < [D_b] - [D_y], v, r >$$

so that

$$||\partial[C_{y,r}]||(\{v \leq r\}) \leq ||\partial[E]||(\{v \leq r\}) + \mathbf{M}(< [D_y] - [D_b], v, r >).$$

It follows that

$$b(y,r) \leq c(y,r) + d(y,r)$$

where for $(y,r) \in Z$ we have set

$$c(y,r) = \epsilon ||\partial[E]||(\{v \leq r\}) + U_y(E)$$

and

$$d(y,r) = U_y(C_{y,r}) - U_y(E) + \mathbf{M}(< [D_y] - [D_b], v, r >).$$

By (4.4.6) we have

$$\int_0^R \mathbf{M}(< [D_y] - [D_b], v, s >)\, d\mathcal{L}^1 s \leq \Sigma_{\{0 < v < R\}}(D_y, D_b)$$

whenever $y \in \mathbb{R} \sim \{0\}$. Suppose $0 < \rho < R$ and let r be a decreasing sequence in (ρ, R) with limit ρ. Suppose η is a sequence of positive real numbers with limit zero. In view of (7.4.7) there is a sequence δ of positive real numbers such that

$$\frac{1}{r_\nu - \rho} \int_\rho^{r_\nu} \mathbf{M}(< [D_y] - [D_b], v, s >)\, d\mathcal{L}^1 s < \eta_\nu \quad \text{provided } 0 < |y - b| < \delta_\nu.$$

for any $\nu \in \mathbf{P}$. Consequently, there are decreasing sequences y in $\mathbb{R} \sim \{b\}$ and s in (ρ, r) with limits b and ρ, respectively, such that $s_\nu < r_\nu$, $(y_\nu, s_\nu) \in W$ and

$$\mathbf{M}(< [D_{y_\nu}] - [D_b], v, s_\nu >) \leq \eta_\nu$$

for any $\nu \in \mathbf{P}$. It follows from (7.4.6) and (7.4.7) that

$$\lim_{\nu \to \infty} d(y_\nu, s_\nu) = 0$$

and from (7.3.2) that

$$\lim_{\nu \to \infty} c(y_\nu, s_\nu) = ||\partial[E]||(\{v \leq \rho\}) + U_b(E).$$

From (4.2.2) and the facts that $a \leq b$ on W and $b \leq c + d$ on Z we infer that

$$\epsilon||\partial[D_b]||(\{v < \rho\}) + U_b(D_b) \leq \liminf_{\nu \to \infty} a(y_\nu, s_\nu) = \epsilon||\partial[E]||(\{v \leq \rho\}) + U_b(E).$$

Owing to the arbitrariness of ρ we find that (7.4.4) holds. □

Theorem 7.4.3. *Suppose $G \in \mathcal{G}(\Omega)$, H is a compact subset of Ω such that*

$$\mathbf{spt}\,[G^+] - [G^-] \subset H \times \mathbb{R}$$

and, for $\mathcal{L}^1$ almost all y,

(7.4.8)
$$\{x : (x, y) \notin G\} \in \mathbf{n}_\epsilon(-U_y) \quad \textit{if } y < 0 \textit{ and } \quad \{x : (x, y) \in G\} \in \mathbf{n}_\epsilon(U_y) \quad \textit{if } y > 0.$$

Then $G^\downarrow \in \mathbf{m}_\epsilon(F)$.

Proof. Note that $\mathbf{spt}\,[G^\downarrow] \subset H$. For each $y \in \mathbb{R} \sim \{0\}$ let

$$D_y = \begin{cases} \{x : (x, y) \in G\} & \text{if } y > 0, \\ \{x : (x, y) \notin G\} & \text{if } y < 0. \end{cases}$$

Let K be a compact subset of Ω and let $g \in \mathcal{F}(\Omega) \cap \mathbf{BV}^{loc}(\Omega)$ be such that $\mathbf{spt}\,[G^\downarrow - g] \subset K$. It follows that $\mathbf{spt}\,[g] \subset H \cup K$.

Let Y be the set of $y \in \mathbb{R} \sim \{0\}$ such that (7.4.8) holds. For $y \in Y \cap (-\infty, 0)$ we have

$$\mathbf{spt}\,[\{g < y\}] - [D_y] \subset H \cup K$$

which, as $D_y \in \mathbf{n}_\epsilon(-U_y)$, implies

$$||\partial[D_y]||(H \cup K) - U_y(D_y) \leq ||\partial[\{g < y\}]||(H \cup K) - U_y(\{g < y\}).$$

For $y \in Y \cap (0, \infty)$ we have

$$\mathbf{spt}\,[\{g \geq y\}] - [D_y] \subset H \cup K$$

which, as $D_y \in \mathbf{n}_\epsilon(U_y)$, implies

$$||\partial[D_y]||(H \cup K) + U_y(D_y) \leq ||\partial[\{g \geq y\}]||(H \cup K) + U_y(\{g \geq y\}).$$

Integrating over $y \in \mathbb{R}$ with respect to $\mathcal{L}^1$ and using Proposition 4.11.4, Theorem 7.4.1, (4.4 and (ii) of Theorem 7.2.1 we find that

$$\begin{aligned}
&||\partial[G^{\downarrow}]||(H \cup K) + F(G^{\downarrow}) \\
&\quad\le \int_{-\infty}^{\infty} ||\partial[D_y]||(H \cup K)\, d\mathcal{L}^1 y + F^{\uparrow}(G) \\
&\quad= \int_{-\infty}^{0} ||\partial[D_y]||(H \cup K) - U_y(D_y)\, d\mathcal{L}^1 y \\
&\qquad + \int_{0}^{\infty} ||\partial[D_y]||(H \cup K) + U_y(D_y)\, d\mathcal{L}^1 y \\
&\quad\le \int_{-\infty}^{0} ||\partial[\{g < y\}]||(H \cup K) - U_y(\{g < y\})\, d\mathcal{L}^1 y \\
&\qquad + \int_{0}^{\infty} ||\partial[\{g \ge y\}]||(H \cup K) + U_y(\{g \ge y\})\, d\mathcal{L}^1 y \\
&\quad= ||\partial[g]||(H \cup K) + F(g).
\end{aligned}$$

□

7.5. **The denoising case, II.** Suppose s, γ, F are as in 1.3. We illustrate the foregoing notions in this case.

Let A be the set of $a \in \Omega$ such that

$$\lim_{r \downarrow 0} r^{-n} \int_{\mathbf{B}^n(a,r)} |s(x) - s(a)|\, d\mathcal{L}^n = 0.$$

Proposition 7.5.1. *We have*

(i) $\mathcal{L}^n(\Omega \sim A) = 0$.
(ii) $k(a, y) = \gamma(y - s(a)) - \gamma(-s(a))$ *whenever* $(a, y) \in A \times \mathbb{R}$.
(iii) F *is convex if and only if* γ *is convex.*

Proof. (i) follows from elementary differentiation theory for $\mathcal{L}^n$ as in [FE, 2.9]. (ii) follows from the uniform continuity of γ on compact sets. From Theorem 7.3.1 we find that F is convex if and only if $\mathbb{R} \ni y \mapsto k(x, y)$ is convex for each $x \in \Omega$ from which (iii) follows. □

For each $y \in \mathbb{R}$ we let

$$\beta_l(y) = \liminf_{z \downarrow y} \frac{\gamma(z) - \gamma(y)}{z - y} \quad \text{and we let} \quad \beta_u(y) = \limsup_{z \downarrow y} \frac{\gamma(z) - \gamma(y)}{z - y}. \tag{7.5.1}$$

Note that β_l, β_u are Borel functions.

Proposition 7.5.2. *For each* $y \in \mathbb{R}$ *we have*

$$L_y(E) = \int_E \beta_l(y - s(x))\, d\mathcal{L}^n x \quad \textit{and} \quad U_y(E) = \int_E \beta_u(y - s(x))\, d\mathcal{L}^n x$$

whenever $E \in \mathcal{M}(\Omega)$*. In particular,* L_y *and* U_y *are local for each* $y \in \mathbb{R}$*.*

Proof. Since γ is locally Lipschitzian the Proposition follows from the Lebesgue Dominated Convergence Theorem. □

Corollary 7.5.1. *Suppose* $1 < p < \infty$ *and* $\gamma(y) = |y|^p/p$ *for* $y \in \mathbb{R}$. *Then*

$$L_y(E) = U_y(E) = \int_{E\cap\{s<y\}} (y - s(x))^{p-1}\, d\mathcal{L}^n x - \int_{E\cap\{s>y\}} (s(x) - y)^{p-1}\, d\mathcal{L}^n x$$

whenever $y \in \mathbb{R} \sim \{0\}$ *and* $E \in \mathcal{M}(\Omega)$.

Proof. Simple calculation. □

7.6. **The Chan-Esedoglu functional.** Let us now suppose

$$\gamma(z) = |z| \quad \text{for } z \in \mathbb{R}$$

and let

$$F(f) = \int_\Omega |f - s|\, d\mathcal{L}^n \quad \text{for } f \in \mathcal{F}(\Omega).$$

Then

$$\beta_l(y) = \begin{cases} -1 & \text{if } -\infty < y \le 0, \\ 1 & \text{if } 0 < y < \infty. \end{cases} \quad \text{and} \quad \beta_u(y) = \begin{cases} -1 & \text{if } -\infty < y < 0, \\ 1 & \text{if } 0 \le y < \infty \end{cases}$$

Suppose $y \in (0, \infty)$. Then

$$\begin{aligned} L_y(E) &= -\mathcal{L}^n(E \cap \{s \ge y\}) + \mathcal{L}^n(E \cap \{s < y\}) \\ &= -\mathcal{L}^n(\{s \ge y\}) + \mathcal{L}^n(\{s \ge y\} \sim E) + \mathcal{L}^n(E \sim \{s \ge y\}) \\ &= N_{\{s\ge y\}}(E) - \mathcal{L}^n(\{s \ge y\}) \end{aligned}$$

and

$$\begin{aligned} U_y(E) &= -\mathcal{L}^n(E \cap \{s > y\}) + \mathcal{L}^n(E \cap \{s \le y\}) \\ &= -\mathcal{L}^n(\{s > y\}) + \mathcal{L}^n(\{s > y\} \sim E) + \mathcal{L}^n(E \sim \{s > y\}) \\ &= N_{\{s>y\}}(E) - \mathcal{L}^n(\{s > y\}) \end{aligned}$$

whenever $E \in \mathcal{M}(\Omega)$.

Suppose $y \in (-\infty, 0)$. Then

$$\begin{aligned} L_y(E) &= -\mathcal{L}^n(E \cap \{s \ge y\}) + \mathcal{L}^n(E \cap \{s < y\}) \\ &= -\mathcal{L}^n(E \sim \{s < y\}) + \mathcal{L}^n(\{s < y\}) - \mathcal{L}^n(\{s < y\} \sim E) \\ &= -N_{\{s<y\}}(E) + \mathcal{L}^n(\{s < y\}) \end{aligned}$$

and

$$\begin{aligned} U_y(E) &= -\mathcal{L}^n(E \cap \{s > y\}) + \mathcal{L}^n(E \cap \{s \le y\}) \\ &= -\mathcal{L}^n(E \sim \{s \le y\}) + \mathcal{L}^n(\{s < y\}) - \mathcal{L}^n(\{s \le y\} \sim E) \\ &= -N_{\{s\le y\}}(E) + \mathcal{L}^n(\{s \le y\}) \end{aligned}$$

whenever $E \in \mathcal{M}(\Omega)$.

This implies

$$\begin{aligned} \mathbf{n}_\epsilon(-L_y) &= \mathbf{n}_\epsilon(N_{\{s<y\}}) \quad \text{if } y < 0; \\ \mathbf{n}_\epsilon(L_y) &= \mathbf{n}_\epsilon(N_{\{s\ge y\}}) \quad \text{if } y > 0; \\ \mathbf{n}_\epsilon(-U_y) &= \mathbf{n}_\epsilon(N_{\{s\le y\}}) \quad \text{if } y < 0; \\ \mathbf{n}_\epsilon(U_y) &= \mathbf{n}_\epsilon(N_{\{s>y\}}) \quad \text{if } y > 0; \end{aligned}$$

8. Curvature and conjugacy.

8.1. **First and second variation.** The following theorem will be proved by calculating the appropriate first and second variations, invoking the Regularity Theorem for $\mathcal{C}_\lambda(\Omega)$ and then utilizing higher regularity results for the minimal surface equation.

Theorem 8.1.1. *Suppose*

(i) $M \in \mathbf{M}(\Omega)$ *and* M *is local;*

(ii) W *is an open subset of* Ω, $k \in \mathbf{N}$, $0 < \mu < 1$, *and* $\zeta | W$ *is of class* $C^{k+\mu}$*; and*

$$M(E) = \int_E \zeta \, d\mathcal{L}^n \quad \text{whenever } E \in \mathcal{M}(\Omega) \text{ and } E \subset W;$$

(iii) $D \in \mathbf{n}_\epsilon(M)$, $S = \mathbf{spt}\,[D]$ *and* $C = W \cap \mathbf{bdry}\, S$*;*

Then C *is and embedded hypersurface of* W *of class* $C^{k+2+\mu}$,

$$\Sigma_W(D, S) = 0$$

and

$$H(x) = -\frac{1}{\epsilon}\zeta(x)\mathbf{n}_S(x) \quad \text{for } x \in C \tag{8.1.1}$$

where H *is the mean curvature vector of* C *and* $\mathbf{n}_S$ *is the outward pointing unit normal to* S *along* C.

Moreover, if ζ *is of class* C^1 *on* W *and* Q *is the square of the length of the second fundamental form of* C *as defined in 3 then*

$$\int_C \epsilon\left(|\nabla_C \phi(x)|^2 + \phi(x)^2 Q(x)\right) - \phi(x)^2 \nabla\zeta(x) \bullet \mathbf{n}_E(x)\, d\mathcal{H}^{n-1}x \geq 0 \tag{8.1.2}$$

for any $\phi \in \mathcal{D}(\Omega)$*; here, for each* $x \in C$, $\nabla_C \phi(x)$ *is the orthogonal projection of* $\nabla\phi(x)$ *on* $\mathbf{Tan}(C, x)$ *and* Q *is the square of the length of the second fundamental form of* C.

Proof. We may assume without loss of generality that $W = \Omega$. For each $x \in \mathbf{b}(D)$ we let $P(x)$ equal to orthogonal projection of $\mathbb{R}^n$ onto $\{v \in \mathbb{R}^n : v \bullet \mathbf{n}_D(x) = 0\}$.

Part One. Suppose $a \in C$, $0 < \mu < 1$ and $0 < \beta < \infty$. From Proposition 6.0.2 and Theorem 5.5.1 there is $r \in (0, \infty)$ such that $\mathbf{U}^n(a, \sqrt{2}r) \subset U$ and $S \in \mathbf{R}(a, r, \mu, \beta)$. Let Ψ, U, g be as in Definition 5.5.1. Let $U = \mathbf{U}^{n-1}(0, r)$ and let $V = \mathbf{U}^1(0, r)$. For each $u \in U$ let

$$G(u) = (u, g(u)), \quad J(u) = \sqrt{1 + |\nabla g(u)|^2}; \quad N(u) = \frac{1}{J(u)}(-\nabla g(u), 1).$$

Note that $\Psi^{-1} \circ G$ carries U diffeomorphically onto $S \cap \Psi^{-1}[U \times \{0\}]$.

Let $j(u, v) = (0, \phi(u))$ for $(u, v) \in U \times V$. Let $(I, h, K) \in \mathcal{V}(\Omega)$ be such that

$$\frac{d}{dt}\Psi \circ h_t \circ \Psi^{-1}(G(u))\big|_{t=0} = j(0, \phi(u)) \quad \text{whenever } u \in U.$$

For each $t \in I$ let $E_t = \{h_t(x) : x \in D\}$, let

$$A(t) = ||\partial[E_t]||(K) \quad \text{and let} \quad B(t) = M(E_t).$$

From Theorems 4.12.1 and 4.12.2 and the fact that D is an (ϵ, M)-minimizer we find that

$$0 = \frac{d}{dt}\epsilon A(t) + B(t)\big|_{t=0} = \int \epsilon\, \mathrm{trace}(a_1(x)) + \zeta(\dot{h}_0 \bullet \mathbf{n}_D)\, d||\partial[D]||$$

where a_1 is as in (4.12.1).

Let $L : \mathbb{R}^n \to \mathbb{R}^{n-1} \times \mathbb{R}$ be such that $L = \partial\Psi(x)$ whenever $x \in \mathbb{R}^n$ and note that L is a linear isometry. For each $u \in U$ let $Q(u)$ be orthogonal projection of $\mathbb{R}^{n-1} \times \mathbb{R}$ onto $\mathbf{Tan}(\mathbf{rng}\, G, G(u))$. If $u \in U$ and $x = \Psi^{-1} \circ G(u)$ then

$$\begin{aligned}
\operatorname{trace} a_1(x) &= \operatorname{trace} \partial\dot{h}_0(x) \circ P(x) \\
&= \operatorname{trace} (L \circ \partial\dot{h}_0(x) \circ L^{-1}) \circ (L \circ P(x) \circ L^{-1}) \\
&= \operatorname{trace} \partial j(G(u)) \circ Q(u) \\
&= \operatorname{trace} \partial j(G(u)) - \partial j(G(u))(N(u)) \bullet N(u) \\
&= -\frac{\nabla g \bullet \nabla \phi}{J^2}(u)
\end{aligned}$$

and

$$\zeta(x)(\dot{h}_0 \bullet \mathbf{n}_S)(x) = \zeta((\Psi^{-1} \circ G)(u)) j(G(u)) \bullet N(u) = \frac{\phi(\zeta \circ (\Psi^{-1} \circ G))}{J}(u).$$

We conclude that

$$\int_U -\frac{\nabla g \bullet \nabla \phi}{J} + \phi(\zeta \circ \Psi^{-1} \circ G)\, d\mathcal{L}^{n-1} = 0.$$

Thus g is a weak solution of the partial differential equation

$$\operatorname{div} J^{-1} \nabla g = -\frac{\zeta \circ \Psi^{-1} \circ G}{\epsilon}.$$

Inasmuch as ∂g is Hölder continuous, standard results on regularity of weak solution of elliptic equations, as found for example in [GT][8.3], imply that g is of class $C^{k+2+\mu}$ and that (8.1.1) holds on $C \cap \Psi^{-1}[U \times V]$.

Since a is an arbitrary point of C we conclude that C is of class $C^{k+2+\mu}$, that C has a second fundamental form and that (8.1.1) holds everywhere on C.

Part Two. We now suppose ζ is continuously differentiable. Let Π, Q, H be as in (3).

Since C is of class C^2 we may choose a function $N : \Omega \to \mathbb{R}^n$ of class C^1 such that $N(b) = \mathbf{n}_D(b)$ whenever $b \in C$. for any $x \in \mathbf{b}(D)$ we have

$$P(x) \circ \Pi(x)(N(x)) \circ P(x) = P(x) \circ \partial N(x) \circ P(x);$$

invoking (8.1.1) we obtain

$$-\frac{\zeta(x)}{\epsilon} = H(x) \bullet N(x) = \operatorname{trace} P(x) \circ \partial N(x) \circ P(x).$$

Next we choose a sequence $Y \in \mathcal{E}(\Omega, \mathbb{R}^n)$ such that

$$|Y_\nu - N| + |\partial Y_\nu - \partial N| \to 0 \quad \text{as } \nu \to \infty$$

uniformly on compact subsets of Ω.

Suppose $\phi \in \mathcal{D}(\Omega)$ and let $K = \mathbf{spt}\, \phi$. Let I be an interval in $\mathbb{R}$ containing 0 such that, for each $\nu \in \mathbf{P}$, if $h_\nu(t, x) = x + \phi(x) Y_\nu(x)$ for $(t, x) \in I \times \Omega$ then $(h_\nu, I, K) \in \mathcal{V}(\Omega)$. Note that

$$\dot{(h_\nu)}_0 = \phi Y_\nu \quad \text{and} \quad \ddot{(h_\nu)}_0 = 0 \quad \text{for all } \nu \in \mathbf{P}.$$

For each $\nu \in \mathbf{P}$ and each $t \in I$ let $E_{\nu,t} = \{h_\nu(t, x) : x \in D\}$, let

$$A_\nu(t) = ||\partial[E_{\nu,t}]||(K) \quad \text{and let} \quad B_\nu(t) = M(E_{\nu,t}).$$

For each $\nu \in \mathbf{P}$ let $a_{1,\nu}, a_{2,\nu}, a_{3,\nu}, A_{1,\nu}, A_{2,\nu}$ equal a_1, a_2, a_3, A_1, A_2, respectively, as in 4.12.1 with h there replaced by h_ν and let b_ν equal b as in 4.12.4 with h there replaced by h_ν. We have that

$$\begin{aligned}
a_{1,\nu} &\to P \circ \partial(\phi N) \circ P = \phi\,(P \circ (\partial N) \circ P); \\
a_{2,\nu} &\to P^\perp \circ \partial(\phi N) \circ P = ((\partial \phi) \circ P)\, N; \\
a_{3,\nu} &\to 0; \\
b_\nu &\to \operatorname{trace}(P \circ \partial(\phi N) \circ P)(\phi N) = (\phi^2 H) N
\end{aligned} \tag{8.1.3}$$

uniformly on $\mathbf{b}(D)$. Thus

$$A_{2,\nu} \to \phi^2 (H \bullet N)^2 + |\partial \phi \circ P|^2 - \phi^2 Q^2 = \phi^2 \frac{\zeta^2}{\epsilon} + |\partial \phi \circ P|^2 - \phi^2 Q^2$$

and

$$B_{2,\nu} \to \phi^2 \zeta (H \bullet N) + \phi^2 (\nabla \zeta \bullet = -\frac{1}{\epsilon} \phi^2 \zeta^2 + \phi^2 (\nabla \zeta \bullet N)$$

uniformly on $\mathbf{b}(D)$. From Theorems 4.12.1 and 4.12.2 and the fact that D is a minimizer we infer that

$$\begin{aligned}
0 &\le \left(\frac{d}{dt}\right)^2 (\epsilon A_\nu + B_\nu)(t)\Big|_{t=0} \\
&= \int \epsilon(\phi^2 \frac{\zeta^2}{\epsilon^2} + |\partial \phi \circ P|^2 - \phi^2 Q^2) - \frac{1}{\epsilon} \phi^2 \zeta^2 + \phi^2 (\nabla \zeta \bullet N)\, d\|\partial[D]\| \\
&= \int \epsilon(|\partial \phi \circ P|^2 - \phi^2 Q^2) + \phi^2 (\nabla \zeta \bullet N)\, d\|\partial[D]\|
\end{aligned}$$

which establishes (8.1.2). □

8.2. **The denoising case, III.** Suppose s, γ and F are as in 1.3; γ is convex; $0 < \epsilon < \infty$; and $f \in \mathbf{m}_\epsilon(F)$.

For each $y \in \mathbb{R} \sim \{0\}$ let

$$D_y = \begin{cases} \{f \ge y\} & \text{if } y > 0, \\ \{f < y\} & \text{if } y < 0. \end{cases}$$

From Theorem 7.4.2 we infer that

$$D \in \begin{cases} \mathbf{n}_\epsilon(-L_y) & \text{if } y < 0, \\ \mathbf{n}_\epsilon(L_y) & \text{if } y > 0. \end{cases}$$

Suppose $y \in \mathbb{R} \sim \{0\}$; W is an open subset of Ω; $\zeta : W \to \mathbb{R}$, k is a nonnegative integer; $0 < \mu < 1$; ζ is of class $C^{k+\mu}$; and

$$\zeta(x) = l_y(x) \quad \text{for } \mathcal{L}^n \text{ almost all } x \in W;$$

and $C = W \cap \mathbf{bdry}\, D_y$. We can then infer the first conclusion of the preceding Theorem and we can infer the second conclusion if, in addition, ζ is of class C^1. *This will trivially be the case if s is essentially constant in U.*

In case $n = 2$, as we shall see in 10 and in [AW2], this is enough to get many interesting results.

Obviously, similar remarks hold if β_l, L_y above are replaced by β_u, U_y.

Let us now assume $n = 2$ and A is a connected component of C.

Suppose *either*

(a) $z \in \mathbb{R}$ and $s(x) = z$ for $\mathcal{L}^2$ almost all $x \in U$;

(b) $\beta(y-z) \neq 0$,

$$R = \frac{\epsilon}{|\beta(y-z)|} \quad \text{and} \quad \sigma = \begin{cases} 1 & \text{if } \beta(y-z) < 0, \\ -1 & \text{if } \beta(y-z) > 0; \end{cases}$$

or

(c) $\gamma(y) = |y|$ for $y \in \mathbb{R}$;
(d) $\mathbf{ess\ sup}\, s|U < y$,

$$R = \epsilon \quad \text{and} \quad \sigma = 1;$$

or

(e) $\gamma(y) = |y|$ for $y \in \mathbb{R}$;
(f) $\mathbf{ess\ inf}\, s|U > y$,

$$R = \epsilon \quad \text{and} \quad \sigma = -1.$$

Then, for some $c \in \mathbb{R}^2$, A is an open arc of the circle $\{x \in \mathbb{R}^2 : |x - c| = R\}$; the length of A does not exceed πR; and, for each $a \in A$, there is $\delta \in (0, R)$ such that

$$E \cap \mathbf{U}^2(a, \delta) = \{x \in \mathbb{R}^2 : |x - c| \leq R\} \cap \mathbf{U}^2(a, \delta) \quad \text{if } \sigma = 1 \tag{8.2.1}$$

and

$$E \cap \mathbf{U}^2(a, \delta) = \{x \in \mathbb{R}^2 : |x - c| \geq R\} \cap \mathbf{U}^2(a, \delta) \quad \text{if } \sigma = -1. \tag{8.2.2}$$

Moreover, if (a) holds and $\beta(y-z) = 0$ then A is contained in a straight line.

This result, together with the regularity theorem for $\mathcal{C}_\lambda(\Omega)$, will allow us to produce the examples at the end of this paper.

All of these assertions, except the assertion that the length of A does not exceed πR, follow directly from Theorem 8 with m there equal to 1 and with any $\mu \in (0, 1)$. Care must be take to ascertain the whether the mean curvature vector of A at $a \in A$ is a positive or negative multiple of $\mathbf{n}_E(a)$.

Let L be the length of A and let R be as above. Note that ζ is constant and that the length of second fundamental form equals $1/R^2$. The second variation formula (8.1.2) implies that

$$\int_0^L \phi'(\sigma)^2 - \frac{1}{R^2}\phi(\sigma)^2 \, d\mathcal{L}^1\sigma \geq 0$$

for all continuously differentiable $\phi : [0, L] \to \mathbb{R}$ which are differentiable on $(0, L)$ and which vanish at 0 and L. Letting

$$\phi(\sigma) = \sin \frac{\pi\sigma}{L} \quad \text{for } \sigma \in [0, L]$$

we infer that $L \leq \pi R$.

9. Some additional results.

9.1. **Calibrations.** We suppose throughout this subsection that

$$f : \Omega \to \mathbb{R},$$

that f is C^2 and that

$$\nabla f(x) \neq 0 \quad \text{whenever } x \in \Omega.$$

We let

$$N(x) = |\nabla f(x)|^{-1} \nabla f(x) \quad \text{whenever } x \in \Omega$$

and we let

$$\omega = N \,\lrcorner\, \mathbf{V}^n;$$

thus ω is a differential $(n-1)$-form on Ω of class C^1.

Note that

$$\mathbf{n}_{\{f\geq y\}}(x) = -N(x) \quad \text{whenever } y \in \mathbb{R},\ x \in \Omega \text{ and } f(x) = y.$$

From (4.8.1) we infer that

$$\int \phi\, d||\partial[E]|| \geq -\int \phi\, \mathbf{n}_E \bullet N\, d||\partial[E]|| = -\partial[E](\phi\omega) \tag{9.1.1}$$

whenever E is a subset of Ω with locally finite perimeter, $\phi \in \mathcal{D}(\Omega)$ and $\phi \geq 0$ with equality if $[E] = [\{f \geq y\}]$ for some $y \in \mathbb{R}$.

From (4.5.2 we have

$$d\omega = \operatorname{div}\, N\mathbf{V}^n;$$

thus

$$H_y = \operatorname{div}\, N \quad \text{on } \{f = y\}$$

where H_y is the mean curvature vector of $\{f = y\}$.

Proposition 9.1.1. *Suppose $\phi \in \mathcal{D}(\Omega)$, K is a compact subset of Ω containing the support of ϕ and $(I, h, K) \in \mathcal{V}(\Omega)$ is such that $h_t(x) = x + t\phi(x)N(x)$ for $(x,t) \in \Omega \times I$. Then*

$$\frac{d}{dt}||\partial[f \circ h_t^{-1}]||(K)\big|_{t=0} = -\int_\Omega \phi \operatorname{div} N\, |\nabla f| d\mathcal{L}^n. \tag{9.1.2}$$

Proof. From (1.1.1) we obtain

$$||\partial[f \circ h_t]||(K) = \int_K |\nabla(f \circ h_t)|\, d\mathcal{L}^n.$$

By a straightforward calculation which we leave to the reader one obtains

$$\frac{d}{dt}|\nabla(f \circ h_t)|\big|_{t=0} = \nabla(\phi|\nabla f|) \bullet N.$$

But

$$\nabla(\phi|\nabla f|) \bullet N = \operatorname{div}(\phi|\nabla f|N) - \phi|\nabla f| \operatorname{div} N.$$

$\square$

Proposition 9.1.2. *Suppose $y \in \mathbb{R}$, $g \in \mathbf{BV}^{loc}(\Omega)$, $\phi \in \mathcal{D}(\Omega)$, $\phi \geq 0$ and*

$$\mathbf{spt}\,[f - g] \subset \mathbf{int}\,\{\phi = 1\}.$$

Then

$$\int \phi\, d||\partial[f]|| - \int \phi\, d||\partial[g]|| \leq \int \left(\int_{\{f<y\leq g\}} \operatorname{div} N\ d\mathcal{L}^n - \int_{\{g<y\leq f\}} \operatorname{div} N\ d\mathcal{L}^n \right) d\mathcal{L}^1 y.$$

Proof. Suppose $y \in \mathbb{R}$. Keeping in mind (4.5.5) we use (9.1.1) to obtain

$$\begin{aligned}
\int \phi\, d||\partial[\{f \geq y\}]|| - \int \phi\, d||\partial[\{g \geq y\}]|| \\
&\leq (\partial[\{g \geq y\}] - \partial[\{f \geq y\}])(\phi\omega) \\
&= ([\{g \geq y\}] - [\{f \geq y\}])(d(\phi\omega)) \\
&= ([\{g \geq y\}] - [\{f \geq y\}])(\operatorname{div} N\, \mathbf{V}^n) \\
&= \int_{\{f<y\leq g\}} \operatorname{div} N\ d\mathcal{L}^n - \int_{\{g<y\leq f\}} \operatorname{div} N\ d\mathcal{L}^n.
\end{aligned}$$

Now integrate this inequality over $y \in \mathbb{R}$ and use (4.9.1). □

Proposition 9.1.3. *Suppose* $0 \le \lambda < \infty$. *We have*

$$f \in \mathcal{B}_\lambda(\Omega) \;\Leftrightarrow\; |\operatorname{div} N\,| \le \lambda.$$

Proof. Suppose $|\operatorname{div} N\,| \le \lambda$. Let $g \in \mathbf{BV}^{loc}(\Omega)$ be such that $\mathbf{spt}\,[f-g]$ is compact and let $\phi \in \mathcal{D}(\Omega)$ be such that $\mathbf{spt}\,[f-g] \subset \mathbf{int}\,\{\phi = 1\}$. From the "layer cake" formula (4.10.2) we infer that

$$\begin{aligned}
&\int \left(\int_{\{f<y\le g\}} - \int_{\{g<y\le f\}} \right) \operatorname{div} N \; d\mathcal{L}^n d\mathcal{L}^1 y \\
&\quad \le \lambda \int \left(\int_\Omega \left| 1_{\{g<y\le f\}} - 1_{\{f<y\le g\}} \right| \, d\mathcal{L}^n \right) d\mathcal{L}^1 y \\
&\quad = \lambda \int_\Omega |f-g| \, d\mathcal{L}^n.
\end{aligned}$$

From the preceding Proposition we infer that $f \in \mathcal{B}_\lambda(\Omega)$.

On the other hand, suppose $f \in \mathcal{B}_\lambda(\Omega)$ and $y \in \mathbf{rng}\, f$. Suppose $\psi \in \mathcal{D}(\Omega)$. From Theorem 5.3.1 we obtain

$$\int \operatorname{trace} P(x) \circ \partial(\psi N)(x) \circ P(x) \, d\|\partial[\{f \ge y\}]\|x \le \lambda \int |\psi| \, d\|\partial[\{f \ge y\}]\|$$

where, for each $x \in \{f = y\}$, we have let $P(x)$ be orthogonal projection of $\mathbb{R}^n$ onto $\{v \in \mathbb{R}^n : v \bullet N(x) = 0\}$. To complete the proof we need only observe that

$$\operatorname{trace} P(x) \circ \partial(\psi N)(x) \circ P(x) = \psi(x) \operatorname{div} N \quad \text{whenever } x \in \{f = y\}.$$

Owing to the arbitrariness of ψ we conclude that $|\operatorname{div} N| \le \lambda|$ □

Proposition 9.1.4. *Suppose* $F \in \mathbf{F}(\Omega)$; *F is local and convex; u is as in Definition 7.2.2; and* $0 < \epsilon < \infty$.

Then $f \in \mathbf{m}_\epsilon(F)$ *if and only if*

$$\operatorname{div} N = \frac{u(x, f(x))}{\epsilon} \quad \textit{for } \mathcal{L}^n \textit{ almost all } x \in \Omega. \tag{9.1.3}$$

Proof. Suppose (9.1.3) holds, $g \in \mathbf{BV}^{loc}(\Omega)$ and $\mathbf{spt}\,[f-g]$ is compact. Recall from Theorem 7.3.1 that

$$\mathbb{R} \ni y \mapsto u(x,y) \quad \text{is nondecreasing}$$

for $\mathcal{L}^n$ almost all $x \in \Omega$. For any $y \in \mathbb{R}$ we estimate

$$\begin{aligned}
&\epsilon \left(\int_{\{f<y\le g\}} - \int_{\{g<y\le f\}} \right) \operatorname{div} N \; d\mathcal{L}^n \\
&\quad = \int_{\{f<y\le g\}} u(x, f(x)) \, d\mathcal{L}^n x - \int_{\{g<y\le f\}} u(x, f(x)) \, d\mathcal{L}^n x \\
&\quad \le \int_{\{f<y\le g\}} u(x, y) \, d\mathcal{L}^n x - \int_{\{g<y\le f\}} u(x, y) \, d\mathcal{L}^n x.
\end{aligned}$$

Moreover, for any $y \in \mathbb{R} \sim \{0\}$ we have

$$\int_{\{f<y\le g\}} u(x,y)\,d\mathcal{L}^n x - \int_{\{g<y\le f\}} u(x,y)\,d\mathcal{L}^n x$$
$$= \begin{cases} U_y(\{f<y\}) - U_y(\{g<y\}) & \text{if } y<0, \\ U_y(\{g\ge y\}) - U_y(\{f \ge y\}) & \text{if } y>0. \end{cases}$$

Integrating with respect to y and invoking Proposition 9.1.2 and Theorem 7.2.1 (v) we infer that $f \in \mathbf{m}_\epsilon(F)$.

To prove the converse statement, suppose $\phi \in \mathcal{D}(\Omega)$, let $(I,h,K) \in \mathcal{V}(\Omega)$ be such that $h_t(x) = x + t\phi(x)N(x)$ for $(t,x) \in I \times K$, let

$$a(t) = ||\partial[f\circ h_t]||(K) \quad \text{and let} \quad b(t) = F(f \circ h_t)$$

whenever $t \in I$. From Proposition 1.1.1 we have

$$\dot{a}(0) = \int_\Omega \phi(x)\,\mathrm{div}\,N\,(x)|\nabla f|(x)\,d\mathcal{L}^n x;$$

moreover, from (iv) of Theorem 7.1.1 and (7.2.3) we obtain

$$\begin{aligned}\dot{b}(0) &= \int_\Omega \frac{d}{dt} k(x, f\circ h_t(x))\big|_{t=0}\,d\mathcal{L}^n x \\ &= \int_\Omega u(x,f(x))\phi(x)\nabla f(x) \bullet N(x)\,d\mathcal{L}^n x \\ &= \int_\Omega u(x,f(x))\phi(x)|\nabla f|(x)\,d\mathcal{L}^n x.\end{aligned}$$

Since $\epsilon a(0) + b(0) \le \epsilon a(t) + b(t)$ for $t \in I$ we infer that $\dot{a}(0) + \dot{b}(0) = 0$. Owing to the arbitrariness of ϕ we infer that (I) holds. □

9.2. **Some results for functionals on sets.** See [AC] for a similar result in a different context.

Proposition 9.2.1. *Suppose $M, N \in \mathbf{M}(\Omega)$, M and N are local, $0<\epsilon<\infty$, $D \in \mathbf{m}_\epsilon(M)$, $E \in \mathbf{m}_\epsilon(N)$ and $\mathbf{spt}\,[D \cup E]$ is compact. Then*

$$\hat{N}(E \sim D) \le \hat{M}(E \sim D).$$

In particular, if

$$\hat{M}(G) < \hat{N}(G) \quad \textit{whenever } G \in \mathcal{M}(\Omega) \textit{ and } \mathcal{L}^n(G) > 0$$

then

$$\mathcal{L}^n(E \sim D) = 0.$$

Proof. Without loss of generality we may assume $M = \hat{M}$ and $N = \hat{N}$. Since $\mathbf{spt}\,[D] \cup \mathbf{spt}\,[E] \subset \mathbf{spt}\,[D \cup E]$ we have

$$\epsilon\mathbf{M}(\partial[D]) + M(D) \le \epsilon\mathbf{M}(\partial[D\cup E]) + M(D \cup E)$$

and

$$\epsilon\mathbf{M}(\partial[E]) + N(E) \le \epsilon\mathbf{M}(\partial[D\cap E]) + N(D \cap E).$$

Also,

$$\begin{aligned}
&\mathbf{M}(\partial[D \cup E]) + \mathbf{M}(\partial[D \cap E]) \\
&\qquad = \int_0^1 \mathbf{M}(\partial[\{1_D + 1_E \geq y\}]\, d\mathcal{L}^1 y + \int_1^2 \mathbf{M}(\partial[\{1_D + 1_E \geq y\}]\, d\mathcal{L}^1 y \\
&\qquad = \mathbf{M}(\partial[1_D + 1_E]) \\
&\qquad \leq \mathbf{M}(\partial[D]) + \mathbf{M}(\partial[E]).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\epsilon(\mathbf{M}(\partial[D]) + \mathbf{M}(\partial[E])) + M(D \sim E) + M(D \cap E) + N(E \sim D) + N(E \cap D) \\
&\quad = \epsilon(\mathbf{M}(\partial[D]) + \mathbf{M}(\partial[E])) + M(D) + N(E) \\
&\quad \leq \epsilon(\mathbf{M}(\partial[D \cap E]) + \mathbf{M}(\partial[D \cup E])) + M(D \cup E) + N(D \cap E) \\
&\quad \leq \epsilon(\mathbf{M}(\partial[D] + \mathbf{M}(\partial[E])) + M(D \cup E) + N(D \cap E) \\
&\quad = \epsilon(\mathbf{M}(\partial[D] + \mathbf{M}(\partial[E])) + M(D \sim E) + M(D \cap E) + M(E \sim D) + N(D \cap E).
\end{aligned}$$

□

I got the idea for following Theorem from [AC]

Theorem 9.2.1. *Suppose $S \in \mathcal{M}(\Omega)$, $0 < \epsilon < \infty$, $\mathcal{A}$ is a nonempty subfamily of $\mathbf{n}_\epsilon(N_S)$ and $\mathbf{spt}\,[\cup\mathcal{A}]$ is compact. Then*

$$\cap\mathcal{A} \in \mathbf{n}_\epsilon(N_S) \quad \textit{and} \quad \cup\mathcal{A} \in \mathbf{n}_\epsilon(N_S).$$

Proof. Let

$$F(f) = \int_\Omega |f - 1_S|\, d\mathcal{L}^n \quad \text{for } f \in \mathcal{F}(\Omega).$$

For each $y \in \mathbb{R}$ let U_y be as in Definition (7.2.2). Recall from that

$$\mathbf{n}_\epsilon(U_y) = \begin{cases} \mathbf{n}_\epsilon(N_\emptyset) & \text{if } 1 \leq y < \infty, \\ \mathbf{n}_\epsilon(N_S) & \text{if } 0 < y < 1, \end{cases} \quad \text{and} \quad \mathbf{n}_\epsilon(-U_y) = \mathbf{n}_\epsilon(N_\emptyset) \text{if } -\infty < y < 0.$$

Suppose $A, B \in \mathbf{n}_\epsilon(N_S)$ and $0 < a < b < c < 1$. Let

$$G = (A \times (0, b)) \cup (B \times (b, 1)) \in \mathcal{G}(\Omega).$$

By Theorem 7.4.3 we find that $G^\downarrow \in \mathbf{m}_\epsilon(F)$. From Theorem 7.4.2 we infer that

$$A \cup B = \{G^\downarrow > a\} \in \mathbf{n}_\epsilon(N_S) \quad \text{and} \quad A \cap B = \{G^\downarrow > c\} \in \mathbf{n}_\epsilon(N_S).$$

It follows that the Theorem holds if $\mathcal{A}$ is finite.

Let

$$\alpha = \sup\{\mathcal{L}^n(A) : A \in \mathcal{A}\},$$

note that $0 \leq \alpha < \infty$ and let B be a sequence in $\mathcal{A}$ such that

$$\lim_{\nu \to \infty} \mathcal{L}^n(B_\nu) = \alpha$$

Let $C_\nu = \cup_{\mu=1}^{\nu} B_\mu$ for each $\nu \in \mathbf{P}$. Then C is a nondecreasing sequence in $\mathcal{A}$. It follows from the result of the preceding paragraph and Proposition 4.9.2 that $D = \cup_{\nu=1}^{\infty} C_\nu \in \mathbf{n}_\epsilon(N_S)$. Obviously, $D \subset \cup\mathcal{A}$. Were it the case that $\mathcal{L}^n(\cup\mathcal{A} \sim D) > 0$ there would exist $E \in \mathcal{A}$ such that $\mathcal{L}^n(E \sim D) > 0$. This would imply

$$\mathcal{L}^n(D \cup E) = \liminf_{\nu \to \infty} \mathcal{L}^n(C_\nu \cup E) > \lim_{\nu \to \infty} \mathcal{L}^n(C_\nu) = \alpha$$

which in turn would imply that $\mathcal{L}^n(D \cup E) > \alpha$ which is impossible. Thus $\mathcal{L}^n(\cup\mathcal{A} \sim D) = 0$ and this implies $\cup\mathcal{A} \in \mathbf{n}_\epsilon(N_S)$.

To handle $\cap\mathcal{A}$ we let

$$\alpha = \inf\{\mathcal{L}^n(A) : A \in \mathcal{A}\},$$

choose a sequence B in $\mathcal{A}$ such that

$$\lim_{\nu\to\infty} \mathcal{L}^n(B_\nu) = \alpha,$$

let $C_\nu = \cap_{\mu=1}^{\nu} B_\mu$ for each $\nu \in \mathbf{P}$, note that C is a nonincreasing sequence in $\mathcal{A}$ and argue that $D = \cap_{\nu=0}^{\infty} C_\nu \in \mathcal{A}$. □

Theorem 9.2.2. *Suppose $M \in \mathbf{M}(\mathbb{R}^n)$; M is local; C is a closed convex subset of $\mathbb{R}^n$ and*

$$M(E) \ge M(\emptyset) \quad \text{whenever } E \in \mathcal{M}(\mathbb{R}^n) \text{ and } \quad \mathcal{L}^n(E \cap C) = 0. \tag{9.2.1}$$

Then $\mathbf{spt}\,[D]$ is compact subset of C whenever $D \in \mathbf{n}_\epsilon(M)$.

Remark 9.2.1. *Evidently, (9.2.1) is equivalent to the statement that $m(x) \ge 0$ for $\mathcal{L}^n$ almost all $x \in \mathbb{R}^n \sim C$ where m is as in Proposition 7.1.2.*

Proof. Suppose $D \in \mathbf{n}_\epsilon(M)$. It follows from Proposition 6.0.2 and Corollary 5.4.1 that $\mathbf{spt}\,[D]$ is compact. From Theorem 4.8.2 we find that

$$\mathbf{M}(\partial[C \cap D]) \le \mathbf{M}(\partial[D]).$$

Moreover, as M is local and $D \in \mathbf{n}_\epsilon(M)$,

$$\epsilon(\mathbf{M}(\partial[D]) - \mathbf{M}(\partial[D \cap C])) \le M(D \cap C) - M(D) = M(\emptyset) - M(D \sim C) \le 0.$$

Thus $\mathbf{M}(\partial[C\cap D]) = \mathbf{M}(\partial[D])$ so the Theorem now follows from Theorem 4.8.2. □

9.3. **Two very useful theorems in the denoising case.** We suppose throughout this subsection that

$$\gamma : \mathbb{R} \to \mathbb{R},$$

γ is locally Lipschitzian, γ is decreasing on $(0, \infty)$ and γ is increasing on $(0, \infty)$. We let

$$F(f) = \int_\Omega \gamma(f(x) - s(x))\, d\mathcal{L}^n x \quad \text{whenever } f \in \mathcal{F}(\Omega).$$

Proposition 9.3.1. *Suppose $0 < \epsilon < \infty$, $f \in \mathbf{m}_\epsilon(F)$,*

$$u = \inf\{\mathbf{ess\ sup}\, f|(\Omega \sim K) : K \textit{ is a compact subset of } \Omega\};$$

and

$$l = \sup\{\mathbf{ess\ inf}\, f|(\Omega \sim K) : K \textit{ is a compact subset of } \Omega\}.$$

Then

$$l \wedge \mathbf{ess\ inf}\, s \le f(x) \le u \vee \mathbf{ess\ sup}\, s \quad \textit{for } \mathcal{L}^n \textit{ almost all } x \in \Omega.$$

Remark 9.3.1. *It follows from Corollary 5.4.2 that $u = 0$ and $l = 0$ if $\Omega = \mathbb{R}^n$.*

Proof. Suppose $u \vee \mathbf{ess\ sup}\, s < M < \infty$. Then $K = \mathbf{spt}\,[f - f \wedge M]$ is a compact subset of Ω so

$$\begin{aligned}
\int_{\{f>M\}} &\gamma(f(x) - s(x)) - \gamma(M - s(x))\, d\mathcal{L}^n x \\
&= F(f) - F(f \wedge M) \\
&\le \epsilon(||\partial[f \wedge M]||(K) - ||\partial[f]||(K)) \\
&= -\int_M^\infty ||\partial[\{f \ge y\}]||(K)\, d\mathcal{L}^1 y \\
&\le 0.
\end{aligned}$$

For $\mathcal{L}^n$ almost all $x \in \Omega$ such that $f(x) > M$ we have

$$f(x) - s(x) > M - s(x) > 0$$

so that, for such x,

$$\gamma(f(x) - s(x)) - \gamma(M - s(x)) > 0.$$

It follows that $\mathcal{L}^n(\{f > M\}) = 0$. Owing to the arbitrariness of M we find that $\mathcal{L}^n(\{f > u \vee \mathbf{ess\ sup}\, s\}) = 0$.

By a similar argument we deduce that $\mathcal{L}^n(\{f < l \vee \mathbf{ess\ inf}\, s\}) = 0$. □

Theorem 9.3.1. *Suppose* $\Omega = \mathbb{R}^n$, $0 < \epsilon < \infty$, $f \in \mathbf{m}_\epsilon(F)$ *and, for each* $y \in \mathbb{R}$,

$$C(y) \text{ equals the closed convex hull of} \begin{cases} \mathbf{spt}\,[\{s \geq y\}] & \text{if } y \geq 0, \\ \mathbf{spt}\,[\{s < y\}] & \text{if } y < 0. \end{cases}$$

Then

$$\mathbf{spt}\,[\{f \geq y\}] \subset C(y) \quad \text{if } y > 0 \quad \text{and} \quad \mathbf{spt}\,[\{f < y\}] \subset C(y) \quad \text{if } y < 0$$

for $\mathcal{L}^1$ *almost all* y.

Proof. suppose $b \in (0, \infty)$. Let

$$g_b = f 1_{\{f<\}} + b 1_{\{f \geq b\} \sim C(b)} + f 1_{\{f \geq b\} \cap C(b)}$$

and note that

$$\{g_b \geq y\} = \begin{cases} \{f \geq y\} & \text{if } y \leq b, \\ \{f \geq y\} \cap C(b) & \text{if } y > b \end{cases}$$

whenever $y \in \mathbb{R}$. It follows from Theorem 4.8.2 that

$$\mathbf{M}(\partial[\{g_b \geq y\}]) \leq \mathbf{M}(\partial[\{f \geq y\}]) \quad \text{whenever } y \in \mathbb{R}.$$

Let $K_b = \mathbf{spt}\,[f - g_b]$ for each $b \in (0, \infty)$. Since $\{f - g_b \neq 0\} \subset \{f > b\}$ we infer from Theorem 6.0.1, Theorem 5.2.6, and Theorem 5.4.1 we infer that K_b is compact. Since $f \in \mathbf{m}_\epsilon(F)$ we infer with the help of (4.5.5) that

$$\begin{aligned} &\int_{\{f>b\} \sim C(b)} \gamma(f(x) - s(x)) - \gamma(b - s(x))\, d\mathcal{L}^n x \\ &\qquad = F(f) - F(g_b) \\ &\qquad \leq \epsilon(||\partial[g_b]||(K_b) - ||\partial[f]||(K_b)) \\ &\qquad = \epsilon \int_b^\infty ||\partial[\{g_b \geq y\}]||(K_b) - ||\partial[\{f \geq y\}]||(K_b)\, d\mathcal{L}^1 y \\ &\qquad \leq 0. \end{aligned}$$

which implies $\mathcal{L}^n(\{f > b\} \sim C(b)) = 0$.

In a similar fashion one handles the case $b < 0$. □

10. Some examples.

Suppose

$$n = 2 \quad \text{and} \quad \Omega = \mathbb{R}^2.$$

Let

$$S = [-1, 1] \times [-1, 1] \in \mathcal{M}(\mathbb{R}^2) \quad \text{and let} \quad s = 1_S \in \mathcal{F}(\mathbb{R}^2).$$

Suppose $1 \leq p < \infty$ and

$$\gamma(y) = \frac{1}{p}|y|^p \quad \text{for } y \in \mathbb{R}$$

and let

$$F(f) = \int \gamma(y - s(x))\, d\mathcal{L}^2 x \quad \text{whenever } f \in \mathcal{M}(\mathbb{R}^2).$$

For each $r \in (0,1]$ and each $i = 0,1,2,3$ let

$$A_{i,r} = \{(1-r, 1-r) + r\{(\cos\theta, \sin\theta) : i\pi/2 \le \theta \le (i+1)\pi/2\}.$$

Let

$$C_r$$

be the convex hull of $\cup_{i=0}^3 A_{i,r}$.

Theorem 10.0.2. *Suppose* $0 < \epsilon < \infty$ *and*

$$T = \{[g] : g \in \mathbf{m}_\epsilon(F)\}.$$

If $(1 + \sqrt{\pi}/2)\epsilon > 1$ *then*

$$T = \{0\}.$$

If $(1 + \sqrt{\pi}/2)\epsilon = 1$ *and* $p = 1$ *then*

$$T = \{t[1_{C_\epsilon}] : 0 \le t \le 1\}.$$

If $(1 + \sqrt{\pi}/2)\epsilon < 1$ *and* $p = 1$ *then*

$$T = \{[1_{C_\epsilon}]\}.$$

If $(1 + \sqrt{\pi}/2)\epsilon = 1$ *and* $p > 1$ *then*

$$T = \{0\}.$$

If $(1 + \sqrt{\pi}/2)\epsilon < 1$; $p > 1$;

$$Y = 1 - \left(1 + \sqrt{\pi}/2)\epsilon\right)^{1/(p-1)};$$

and $f \in \mathbf{BV}^{loc}(\mathbb{R}^2)$ *is such that*

$$\mathbf{ess\ inf}\, f = 0; \quad \mathbf{ess\ sup}\, f = Y;$$

and, whenever $0 < y < Y$,

$$[\{f \ge y\}] = [1_{C_{r(y)}}]$$

where

$$r(y) = \frac{\epsilon}{(1-y)^{p-1}}$$

then

$$T = \{[f]\}.$$

Proof. For each $y \in \mathbb{R} \sim \{0\}$ let U_y be as in 1.5. We will make use of 7.4.2 and 7.4.3. For this purpose let

$$Q_y = \{[D] : D \in \mathbf{n}_\epsilon(U_y)\} \quad \text{whenever } y \in \mathbb{R} \sim \{0\}.$$

Suppose $E \in \mathcal{M}(\Omega)$ and $\mathcal{L}^2(E) > 0$. From Corollary 7.5.1 and (7.6) we find that $U_y(E) > 0$ if $y \ge 1$ and $-U_y(E) > 0$ if $y < 0$. It follows that

$$Q_y = \{[\emptyset]\} \quad \text{if } y \ge 1 \text{ or } y < 0.$$

Suppose $0 < y < 1$, let

$$Z = \begin{cases} 1 & \text{if } p = 1, \\ (1-y)^{p-1} & \text{if } p > 1 \end{cases} \quad \text{and let} \quad R = \frac{\epsilon}{Z}.$$

Suppose $R \le 1$ and let

$$I = \epsilon \mathbf{M}(\partial [C_R]) + U_y(C_R).$$

We calculate

$$\epsilon \mathbf{M}(\partial [C_R]) = \epsilon(4(2-2R) + 2\pi R$$

and, with the help of Corollary 7.5.1 and (7.6),

$$U_y(C_R) = -Z\mathcal{L}^2(C_R) = -Z(4-(4-\pi)R^2)$$

so

$$\begin{aligned} I &= \epsilon(4(2-2R)+2\pi R) - Z(4-(4-\pi)R^2) \\ &= \frac{-4Z^2 + 8\epsilon Z + (\pi-4)\epsilon^2}{Z} \\ &= -4\frac{(Z-(1+\sqrt{\pi}/2)\epsilon)(Z-(1-\sqrt{\pi}/2)\epsilon)}{Z}. \end{aligned}$$

Since $R \le 1$ we have

$$Z = \epsilon/R \ge \epsilon > (1-\sqrt{\pi}/2)\epsilon.$$

Thus

$$I \le 0 = \mathbf{M}(\partial[\emptyset] + U_y(\emptyset) \;\Leftrightarrow\; Z \ge (1+\sqrt{\pi}/2)\epsilon.$$

Suppose $E \in \mathbf{n}_\epsilon(U_y)$ and $[E] \ne 0$. I claim that

$$R \le 1 \quad \text{and} \quad [E] = \{[C_R]\}. \tag{10.0.1}$$

From Theorem 9.2.2 we infer that $\mathbf{spt}\,[E] \subset S$. Let U equal the interior of S and let $M = U \cap \mathbf{bdry}\,E$. Then $U \cap M \ne \emptyset$ since otherwise we would have $E = S$ in which case M would have corners which is incompatible with Theorem 5.5.1. Let A be a connected component of M. We infer from 8.2 that A is an arc of a circle of radius R the length of which does not exceed πR. Because M can have no corners we find that A meets the interior of the boundary of S tangentially. Thus (10.0.1) holds.

The Theorem now follows from 7.4.2 and 7.4.3. □

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Department of Mathematics, Duke University, Durham, NC 27708-0320
E-mail address: wka@math.duke.edu